Strings in curved space-time

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- The features of the space-time geometry (connection, torsion and nonmetricity)
  - B. Sazdović
    - Torsion and nonmetricity in the stringy geometry (*hep-th/0304086*)
    - Bosonic string theory in background fields by canonical methods *IJMP* **A 20** (2005) 5501;
  - D. S. Popović and B. Sazdović

    The geometrical form for the string space-time action

    Eur. Phys. J. (2007) (*hep-th/ 0701264*)
- Decreasing the number of the Dp-brane dimensions
  - B. Nikolić and B. Sazdović
    - Gauge symmetries decrease the number of Dp-brane dimensions *Phys. Rev.* **D 74** (2006) 045024 (*hep-th/ 0604129*)
    - Gauge symmetries decrease the number of Dp-brane dimensions, II (Inclusion of the Liouville term) (*hep-th/ 0611191*)
- Symmetries of the space-time field equations

Definition of the model

- Action

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu} + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi R^{(2)} \right\}$$

  - ξ^α $(\alpha = 0, 1)$ world-sheet coordinates
  - $x^\mu(\xi)$ $(\mu = 0, 1, ..., D-1)$ space-time coordinates
  - $x^i(\xi)$ $(i = 0, 1, ..., p)$ Dp-brane coordinates
  - String propagates in background defined by
    * metric tensor $G_{\mu\nu}(x)$
    * antisymmetric tensor field $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$
    * dilaton field $\Phi(x)$

- Space-time field equations (Consequence of the quantum world-sheet conformal invariance)

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2D_\mu a_\nu = 0$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2a_\rho B^\rho{}_{\mu\nu} = 0$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = 0$$

$B_{\mu\rho\sigma} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ field strength

$a_\mu = \partial_\mu \Phi$ gradient of dilaton

Canonical analysis

- Currents

$$J_{\pm\mu} = P^{T}{}_{\mu}{}^{\nu} j_{\pm\nu} + \frac{a_\mu}{2a^2} i^{\Phi}_{\pm} = j_{\pm\mu} - \frac{a_\mu}{a^2} j$$

$$i^{F}_{\pm} = \frac{a^\mu}{a^2} j_{\pm\mu} - \frac{1}{2a^2} i^{\Phi}_{\pm} \pm 2\kappa F' , \qquad i^{\Phi}_{\pm} = \pi_F \pm 2\kappa \Phi'$$

where

$$j_{\pm\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu} x^{\nu\prime} , \qquad \Pi_{\pm\mu\nu} \equiv B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$$

$$j = a^\mu j_{\pm\mu} - \frac{1}{2} i^{\Phi}_{\pm} = a^2 (i^{F}_{\pm} \mp 2\kappa F')$$

π_μ and π_F are canonically conjugate momenta to the variables x^μ and F

- Canonical Hamiltonian and energy momentum tensor

$$\mathcal{H}_c = h^{-} T_{-} + h^{+} T_{+}$$

$$T_{\pm} = \mp \frac{1}{4\kappa} \left(G^{\mu\nu} J_{\pm\mu} J_{\pm\nu} + i^{F}_{\pm} i^{\Phi}_{\pm} \right) + \frac{1}{2} i^{\Phi\prime}_{\pm}$$

$$= \mp \frac{1}{4\kappa} \left(G^{\mu\nu} j_{\pm\mu} j_{\pm\nu} - \frac{j^2}{a^2} \right) + \frac{1}{2} (i^{\Phi\prime}_{\pm} - F' i^{\Phi}_{\pm})$$

- Two independent copies of Virasoro algebras

$$\{ T_{\pm}(\sigma), T_{\pm}(\bar\sigma) \} = -[T_{\pm}(\sigma) + T_{\pm}(\bar\sigma)] \delta'(\sigma - \bar\sigma)$$

Equations of motion

-

$$[J^\mu] \equiv \nabla_\mp \partial_\pm x^\mu + {}^\star\Gamma^\mu_{\mp\rho\sigma} \partial_\pm x^\rho \partial_\mp x^\sigma = 0$$

$$[h^\pm] \equiv G_{\mu\nu} \partial_\pm x^\mu \partial_\pm x^\nu - 2\nabla_\pm \partial_\pm \Phi = 0$$

$$[i^F] \equiv R^{(2)} + \frac{2}{a^2}(D_{\mp\mu} a_\nu) \partial_\pm x^\nu \partial_\mp x^\mu = 0$$

- Generalized connection

$${}^\star\Gamma^\rho_{\pm\nu\mu} = \Gamma^\rho_{\pm\nu\mu} + \frac{a^\rho}{a^2} D_{\pm\mu} a_\nu = \Gamma^\rho_{\nu\mu} \pm P^{T\rho}{}_\sigma B^\sigma_{\nu\mu} + \frac{a^\rho}{a^2} D_\mu a_\nu$$

  Under space-time general coordinate transformations the expression ${}^\star\Gamma^\rho_{\pm\nu\mu}$ transforms as a connection.

- The covariant derivatives with respect to the Christoffel connection $\Gamma^\rho_{\nu\mu}$ and to the connection $\Gamma^\rho_{\pm\nu\mu} = \Gamma^\rho_{\nu\mu} \pm B^\rho_{\nu\mu}$, we respectively denote as D_μ and $D_{\pm\mu}$

Geometry of space-time seen by the probe string

- Parallel transport
  $V^\mu(x) \to {}^\circ V^\mu_{\|}(x+dx) = V^\mu + {}^\circ\delta V^\mu$

$$ {}^\circ\delta V^\mu = -{}^\circ\Gamma^\mu_{\rho\sigma} V^\rho dx^\sigma $$

  ${}^\circ\Gamma^\mu_{\rho\sigma}$ affine linear connection

- The covariant derivative

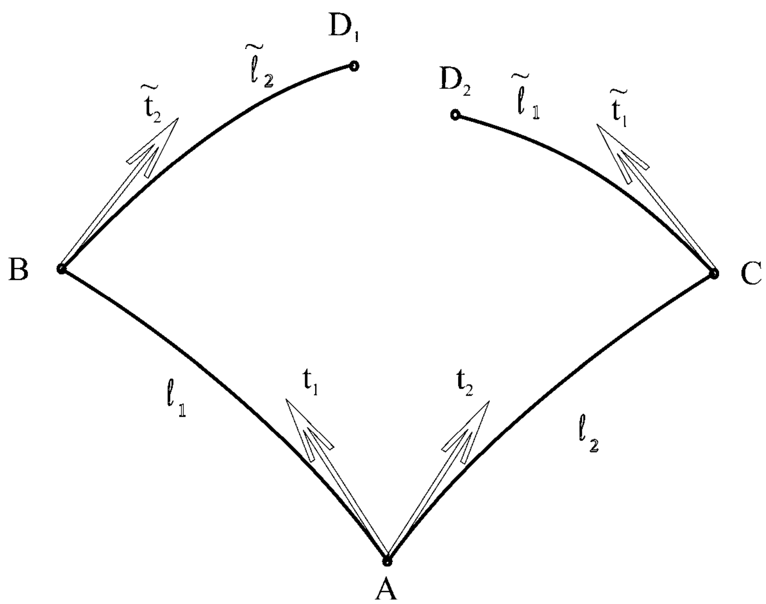
$$ \begin{aligned} {}^\circ DV^\mu &= V^\mu(x+dx) - {}^\circ V^\mu_{\|} = dV^\mu - {}^\circ\delta V^\mu \\ &= (\partial_\nu V^\mu + {}^\circ\Gamma^\mu_{\rho\nu} V^\rho) dx^\nu \equiv {}^\circ D_\nu V^\mu dx^\nu \end{aligned} $$

Torsion

- **torsion** (antisymmetric part of the affine connection)

$$^\circ T^\rho_{\mu\nu} = {}^\circ\Gamma^\rho_{\mu\nu} - {}^\circ\Gamma^\rho_{\nu\mu}$$

Geometrical interpretation: measures the non-closure of the curved "parallelogram"



$$x^\mu(D_2) - x^\mu(D_1) = {}^\circ T^\mu{}_{\rho\sigma}\, t_1^\rho\, t_2^\sigma\, d\ell_1\, d\ell_2$$

Non-metricity

- Covariant derivative is responsible for the comparison of the vectors from different points

- What variable is responsible for comparison of the lengths of the vectors?
  - $V^\mu(x)$: using invariance of the scalar product under the parallel transport
    $V^2(x) = G_{\mu\nu}(x)V^\mu(x)V^\nu(x)$
    $= [G_{\mu\nu}(x) + {}^\circ\delta G_{\mu\nu}(x)]{}^\circ V_{||}^\mu \, {}^\circ V_{||}^\nu$
  - its parallel transport to the point $x + dx$, ${}^\circ V_{||}^\mu$,

$$ {}^\circ V_{||}^2(x+dx) = G_{\mu\nu}(x+dx){}^\circ V_{||}^\mu \, {}^\circ V_{||}^\nu $$

  - Difference of the squares of the vectors

$$ {}^\circ\delta V^2 = {}^\circ V_{||}^2(x+dx) - V^2(x) $$

$$ = [G_{\mu\nu}(x+dx) - G_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)] \, {}^\circ V_{||}^\mu \, {}^\circ V_{||}^\nu $$

    Up to the higher order terms we have

$$ {}^\circ\delta V^2 = [dG_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)]V^\mu V^\nu $$

$$ = {}^\circ DG_{\mu\nu}V^\mu V^\nu \equiv -dx^\rho \, {}^\circ Q_{\rho\mu\nu}V^\mu V^\nu $$

- **nonmetricity** covariant derivative of the metric tensor

$$ {}^\circ Q_{\mu\rho\sigma} = -{}^\circ D_\mu G_{\rho\sigma} $$

Stringy torsion and nonmetricity

- **Stringy torsion** antisymmetric part of the stringy connection

$$^\star T_{\pm}{}^{\rho}_{\mu\nu} = {}^\star\Gamma_{\pm}{}^{\rho}_{\mu\nu} - {}^\star\Gamma_{\pm}{}^{\rho}_{\nu\mu} = \pm 2 P^{T\rho}{}_{\sigma} B^{\sigma}_{\mu\nu}$$

- **Stringy nonmetricity** measures non-compatibility of the metric $G_{\mu\nu}$ with the stringy connection ${}^\star\Gamma^{\mu}_{\pm\nu\rho}$

$$^\star Q_{\pm\mu\rho\sigma} \equiv -{}^\star D_{\pm\mu} G_{\rho\sigma} = \frac{1}{a^2} D_{\pm\mu}(a_\rho a_\sigma)$$

The presence of the dilaton field Φ leads to breaking of the space-time metric postulate

- **Stringy Weyl vector**

$$^\star q_\mu = \frac{1}{D} G^{\rho\sigma\star} Q_{\pm\mu\rho\sigma} = \frac{-4}{D}\partial_\mu \varphi$$

is a gradient of new scalar field φ, defined by the expression

$$\varphi = -\frac{1}{4}\ln a^2 = -\frac{1}{4}\ln(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi)$$

The space-time action

$$S = \int dx \sqrt{-G} e^{-2\Phi} [R - \frac{1}{12} B^2 + 2 D^2 \Phi]$$

$B^2 = B_{\mu\nu\rho} B^{\mu\nu\rho} \qquad D^2\Phi = G^{\mu\nu} D_\mu \partial_\nu \Phi$
is reproduce by the action

$$^\star S = \int d^D x \ \sqrt{-G} e^{-2\varphi} \ ^\star\mathcal{L}$$

$$^\star\mathcal{L} \equiv \, ^\star R + \frac{1}{48} \left(11 \, ^\star T^2 - 26 \, ^\star Q^2 \right) + \frac{1}{3} \left(\frac{5D}{4} \right) {^\star q}^2$$

If the condition $\varphi = \Phi \Longleftrightarrow G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - e^{-4\Phi} = 0$
is satisfied, up to the term with factor $\frac{1}{a^2} \qquad S = \, ^\star S$

Invariant measure

1.Invariant under space-time general coordinate transformations.

2. Preserved under parallel transport $\Longleftrightarrow \, ^\star D_{\pm\mu} {^\star\Omega} = 0.$

3. Enable integration by parts, which can be achieved with help of of the Leibniz rule and the relation

$$\int d^D x \, ^\star\Omega \, ^\star D_{\pm\mu} V^\mu = \int d^D x \partial_\mu (^\star\Omega V^\mu)$$

so that we are able to use Stoke's theorem.

Decreasing the number of the Dp-brane dimensions Open string boundary conditions

- According to the **action principle**, evolution from given initial configuration (at τ_i) to given final configuration (at τ_f) is such that the action is stationary

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \left(\dot{p}_\mu + \gamma'_\mu\right) \delta x^\mu + \int_{\tau_i}^{\tau_f} d\tau (\gamma_\mu \delta x^\mu)/_{\sigma=0}^{\sigma=\pi} = 0$$

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \qquad \gamma_\mu \equiv \frac{\partial \mathcal{L}}{\partial x'^\mu}$$

  - For all variations δx^μ → EM $\qquad \dot{p}_\mu + \gamma'_\mu = 0$
  - At the string endpoints → BC $\qquad (\gamma_\mu \delta x^\mu)/_{\sigma=0}^{\sigma=\pi} = 0$

- Closed string: do not have endpoints → no BC

- Open string
  - *arbitrary variations δx^μ on the string endpoints*
    → **Neumann** boundary conditions

$$\gamma_\mu/_{\sigma=0} = 0, \qquad \gamma_\mu/_{\sigma=\pi} = 0$$

  - *fixed edges of the string*
    → **Dirichlet** boundary conditions

$$\delta x^\mu|_{\sigma=0} = 0\,, \qquad \delta x^\mu|_{\sigma=\pi} = 0$$

Dp-brane

- – $x^i = \{x^0, x^1, ...x^p\}$ *Dp tangential coordinates*
  NN Boundary conditions: $\gamma_i/_{\sigma=\sigma^*} = 0$
  – $x^a = \{x^{p+1}, x^{p+2}, ...x^{D-1}\}$ *Dp normal coordinates*
  DD Boundary conditions: $\delta x^i|_{\sigma^*} = 0$

- Example: $D = 3$
  $x^i = \{x^0 \equiv X, x^1 \equiv Z\}$, $x^a \equiv \{x^3 = Y = 0\}$

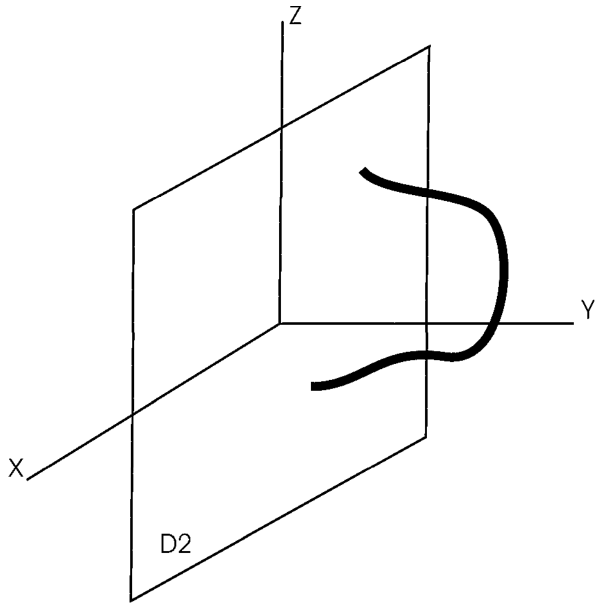


Figure 1: D2-brane stretched over (X,Z) plane

- String endpoints are attached to Dp-brane and move freely on (x^0, x^1) plane

Linear dilaton background

- – Open string propagates in **linear dilaton background**
    - ∗ metric tensor $G_{\mu\nu} = const$
    - ∗ antisymmetric tensor field $B_{\mu\nu} = -B_{\nu\mu} = const$
    - ∗ dilaton field $\Phi = \Phi_0 + a_\mu x^\mu\,, \quad (a_\mu = const)$
  - – Chose for simplicity
    - ∗ $B_{\mu\nu} \to B_{ij}$ and $a_\mu \to a_i$
    - ∗ $G_{\mu\nu} = 0$ for $\mu = i \in \{0, 1, .., p\}$ $\nu = a \in \{p+1, ..., D-1\}$

- Conformal gauge, $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$

$$S = \kappa \int_\Sigma d^2\xi \left\{ \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + 2\eta^{\alpha\beta} a_i \partial_\alpha x^i \partial_\beta F \right\}$$

Action depends on $F \longrightarrow$ no conformal invariance

Non-commutativity in absence of diaton field, $\Phi = 0$

- For $\Phi = 0$, action is F independent

- – Consider $\gamma_i|_{\sigma=\sigma^*}$ as primary constraints
  – Apply Dirac consistency procedure $\gamma_i|_{\sigma=\sigma^*} \to \Gamma_i(\sigma)$

- – Algebra of constraints

$$\{\Gamma_i(\sigma), \Gamma_j(\bar{\sigma})\} = -\kappa G^{eff}_{ij} \delta'(\sigma - \bar{\sigma})$$

  $G^{eff}_{ij} \equiv G_{ij} - 4B_{ik}G^{kq}B_{qj}$ **effective metric tensor**
  – We will refer to it as the **open string metric tensor**, the metric tensor seen by the open string

- Conventions: $(G^{-1}_{eff})^{ij}$ inverse of G^{eff}_{ij}
  $\tilde{V}^i = (G^{-1}_{eff})^{ij} V_j \qquad \tilde{V}^2 = (G^{-1}_{eff})^{ij} V_i V_j$
  $V^i = G^{ij} V_j \qquad V^2 = G^{ij} V_i V_j$

- – We solve SCC constraints $\Gamma_i = 0$

$$x^i(\sigma) = q^i(\sigma) - 2\Theta^{ij} \int_0^\sigma d\sigma_1 p_j(\sigma_1)$$

$$q^i(\sigma) = \frac{1}{2}\left[x^i(\sigma) + x^i(-\sigma)\right], \quad p_i(\sigma) = \frac{1}{2}\left[\pi_i(\sigma) + \pi_i(-\sigma)\right]$$

  – All Dp-brane coordinates are non-commutative

$$\{x^i(\sigma), x^j(\bar{\sigma})\} = 2\Theta^{ij}\theta(\sigma+\bar{\sigma}), \quad \Theta^{ij} = -\frac{1}{\kappa}(G^{-1}_{eff} B G^{-1})^{ij}$$

Linear dilaton and role of field F in open string theory

- – In linear dilaton background $\Phi = \Phi_0 + a_i x^i$, classical theory is F dependent
  – Closed string theory: on the quantum level the field F decouples *(conformal invariance condition)*
  – Open string theory: *Whether field F decouples?*

- – Conformal part of the world-sheet metric $F \to$ dynamical variable
  – $F \to$ **Neumann** boundary conditions
  arbitrary variations δF on the string endpoints

$$\gamma/_{\sigma=0} = 0, \qquad \gamma/_{\sigma=\pi} = 0 \qquad \left(\gamma \equiv \frac{\partial \mathcal{L}}{\partial F'}\right)$$

  – Consider $\gamma_i|_{\sigma=\sigma^*}$ and $\gamma|_{\sigma=\sigma^*}$ as primary constraints
  – Dirac consistency procedure
  $\gamma_i|_{\sigma=\sigma^*} \to \Gamma_i(\sigma)\,, \qquad \gamma|_{\sigma=\sigma^*} \to \Gamma(\sigma)$

- 1. $a^2 \neq 0$ and $\tilde{a}^2 \neq 0$
  – turns coordinate $x_c = a_i x^i$ to **commutative** one
  – F becomes a **noncommutative** variable
  – Field F breaks conformal invariance of open string theory
  Sazdović, Eur. Phys. J. C44 (2005) 599, *hep-th/0408131*

Origins of gauge symmetries

- 2. $a^2 \equiv G^{ij} a_i a_j = 0$ and $\tilde{a}^2 \neq 0$

$$j \equiv a^i \pi_i - \frac{1}{2}\pi + 2\kappa a^i B_{ij} x^{j'} = 2\kappa a^2 \dot{F} ,$$

  Standard canonical constraint

- 3. $a^2 \neq 0$ and $\tilde{a}^2 \equiv (G_{eff}^{-1})^{ij} a_i a_j = 0$
  - Complete set of the constraints $\chi_A = (\Gamma_i, \Gamma)$

$$\{\chi_A(\sigma), \chi_B(\overline{\sigma})\} = -\kappa M_{AB} \delta' ,$$

$$M_{AB} = \begin{pmatrix} \tilde{G}_{ij} & 2a_i \\ 2a_j & 0 \end{pmatrix} .$$

$$\tilde{G}_{ij} = G_{ij} - 4B_{ik}(P_T^0)^{kl} B_{lj} = (\hat{P}_T^1 G_{eff})_{ij}$$

$$\det M_{AB} = -4\tilde{a}^2 \det \tilde{G}_{ij} ,$$

  - Two zeros in $\tilde{a}^2 = 0$ $\rightarrow$ two first class constraints
  - Some of the constraints originated from the boundary conditions **turn from the second class into the first class constraints**

Effective theory

- – First class constraints generate local symmetries
  – After gauge fixing, the gauge conditions and the first class constraints can be treated as second class constraints
  – We solve them together with the true second class constraints

- The original variables in terms of new ones

$$x^i_{Dp}(\sigma) = Q^i(\sigma) - 2\Theta^{ij}\int_0^\sigma d\sigma_1 P_j(\sigma_1)\,,$$

$$(aB)_i x^i/_{\sigma=0,\pi} = 0\,, \qquad F = 0$$

- – Effective variables: $Q^i = (P_{Dp})^i{}_j q^j$, $P_i = (P_{Dp})_i{}^j p_j$

  – Q^i and P_i are canonical variables in the Dp-brane subspace defined by projector P_{Dp}

$$\left\{Q^i(\sigma), P_j(\overline{\sigma})\right\} = (P_{Dp})^i{}_j \delta_s(\sigma,\overline{\sigma})$$

- Noncommutativity tensor

$$\Theta^{ij} = -\frac{1}{\kappa}(G^{-1}_{eff} P_{Dp} B G^{-1} P_{Dp})^{ij} \qquad (\Theta^{ij} = -\Theta^{ji})$$

- Projectors $\quad (\Pi_1)_i{}^j = \frac{n_i \tilde{n}^j}{\tilde{n}^2} \qquad (n_1)_i = (aB)_i$

  $(\Pi^1_T)_i{}^j = \delta_i{}^j - (\Pi_1)_i{}^j \qquad (P_{Dp})^i{}_j = (\Pi^1_T)^i{}_j/_{a^2_*=0}$

Solution of boundary conditions-General features

- – Components (x_1, F) satisfy Dirichlet boundary conditions
  – First class constraints turn some Neuman to Dirichlet boundary conditions and
  number of Dp-brane dimensions decreases

- – **Commutative** degrees of freedom $x_c = a_i x^i_{Dp}$
  – $a_i \Theta^{ij} = 0$, so the effective momentum disappears
  – *Closed string components contain only the open string coordinates, while the momenta are absent*

- **Noncommutativity**
  The closed string components $(x_{nc})^i$*, contains both the open string coordinates and momenta*
  – Separate the center of mass
  $x^i_{Dp}(\sigma) = (x^i_{Dp})_{cm} + X^i_{Dp}(\sigma)$
  – Poisson brackets between the coordinates

$$\{X^i_{Dp}(\tau,\sigma), X^j_{Dp}(\tau,\overline{\sigma})\} = \Theta^{ij} \begin{cases} -1 & \sigma = 0 = \bar{\sigma} \\ 1 & \sigma = \pi = \bar{\sigma} \\ 0 & \text{otherwise} \end{cases} .$$

Conclusion

- Dp-brane features

	fcc	scc	D_{Dp}	(P_{D_p})	DBC	$(x_{nc})^i$	x_c
1	0	p+2	p+2	$\delta_i{}^j$	-	$(\Pi^0_T x)^i\,, F$	x_0
2	1	p+2	p	$(P^1_T)_i{}^j$	$x_1\,, F$	$(P_T x)^i$	x_0
3	2	p	p	$(\hat{P}^1_T)_i{}^j$	$x_1\,, F$	$(\hat{P}_T x)^i$	x_0

$$x_0 = a_i x^i\,, \quad x_1 = (aB)_i x^i$$

$$(P^1_T)_i{}^j = (\Pi^1_T)_i{}^j \Big|_{a^2=0}\,, \qquad (\hat{P}^1_T)_i{}^j = (\Pi^1_T)_i{}^j \Big|_{\tilde{a}^2=0}$$

– In cases 2. and 3. field F disappears, $(F = 0)$
– Additional, open string conformally invariant conditions
$a^2 = 0$ or $\tilde{a}^2 = 0$
Nikolić and Sazdović
Gauge symmetries decrease the number of Dp-brane dimensions
Phys. Rev. **D 74** (2006) 045024 (*hep-th/ 0604129*)

- In all cases
  – effective energy-momentum tensor satisfies Virasoro algebra

$$\{T_\pm, T_\pm\} = -\left[T_\pm(\sigma) + T_\pm(\overline{\sigma})\right]\delta'\,, \quad \{T_\pm, T_\mp\} = 0\,,$$

  – but in **new background**

$$G_{ij} \to (P_{D_p} G_{eff})_{ij}\,, \quad B_{ij} \to 0\,, \quad \Phi \to \Phi_0 + a_i Q^i$$

Conformal invariance with the help of Liouville action

- – New conditions for quantum conformal invariance
  $\beta^G_{\mu\nu} = 0$, $\beta^B_{\mu\nu} = 0 \Longrightarrow \beta^\Phi = c.$
  – Non-linear sigma model becomes conformal field theory
  – There exists a Virasoro algebra with central charge c

- – The remaining anomaly, can be cancelled by introducing Liouville action

$$S_L = -\frac{\beta^\Phi}{2(4\pi)^2\kappa} \int_\Sigma d^2\xi \sqrt{-g} R^{(2)} \frac{1}{\Delta} R^{(2)}$$

  – Advantages
    * Conformal invariance in the presence of field F
    * Noncommutativity parameter depend on central charge c

Realization

- Conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta} \quad \Longrightarrow R^{(2)} = -2\Delta F$

  $$S = \kappa \int_\Sigma d^2\xi \left[\ldots + 2\eta^{\alpha\beta} a_i \partial_\alpha x^i \partial_\beta F + \frac{2}{\alpha}\eta^{\alpha\beta}\partial_\alpha F \partial_\beta F \right]$$

  $\frac{1}{\alpha} = \frac{\beta^\Phi}{(4\pi\kappa)^2}$

  F is a dynamical variable with the Liouville action as a kinetic term

- To cancel the term linear in F, we change the variables, $F \to {}^\star F = F + \frac{\alpha}{2} a_i x^i$

  $$S = \kappa \int_\Sigma d^2\xi \left[\left(\frac{1}{2}\eta^{\alpha\beta}\, {}^\star G_{ij} + \epsilon^{\alpha\beta} B_{ij} \right) \partial_\alpha x^i \partial_\beta x^j + \frac{2}{\alpha}\eta^{\alpha\beta}\partial_\alpha {}^\star F \partial_\beta {}^\star F \right]$$

  - Standard form of the action without dilaton term
  - Redefined Liouville term, $F \to {}^\star F$ (${}^\star F$ decouples)
  - Redefined space-time metric, ${}^\star G_{ij} = G_{ij} - \alpha a_i a_j$
  - Dilaton dependence is through the metric ${}^\star G_{ij}$

- ${}^\star G_{ij}$ and the corresponding effective one ${}^\star G^{eff}_{ij}$ are singular

  $\alpha a^2 = 1 \quad \Longrightarrow \quad \det {}^\star G_{ij} = 0$

  $\alpha \tilde{a}^2 = 1 \quad \Longrightarrow \quad \det {}^\star G^{eff}_{ij} = 0$

  They produce gauge symmetries of the theory

- We choose Neumann boundary conditions for ${}^\star F$
- Zero central charge limit $\alpha \to \infty \quad \Longleftrightarrow \quad \beta^\Phi = c = 0$

Conclusion (in presence of Liouville term)

- We carefully investigate all three cases with Liouville action
  All important results as:
  - Form of the solution
  - Dimensions of Dp-brane
  - Number of commutative and noncommutative variables
  - Noncommutativity relation

  are the same as in the absence of Liouville action

- **Differences** produced by Lioville term
  - Local gauge symmetries appear for $a^2 = \frac{1}{\alpha}$ and $\tilde{a}^2 = \frac{1}{\alpha}$
    instead for $a^2 = 0$ and $\tilde{a}^2 = 0$
  - Variables change the roles , $x_0 \to {}^\star F$ and $F \to x_0$
    * Instead of x_0, the variable ${}^\star F$ is commutative
    * In the first case x_0 is noncommutative instead of F
    * In the second and third case x_0 satisfies the Dirichlet boundary conditions instead of F

Symmetries of the space-time field equations

- World-sheet metric tensor $g_{\alpha\beta}$, in terms of the light-cone variables (h^+, h^-, F)

$$g_{\alpha\beta} = \frac{1}{2}e^{2F}\begin{pmatrix} -2h^-h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix}$$

- – World-sheet Diffeomorphisms $\xi^\mu \to \xi^\mu + \varepsilon^\mu(\xi)$

$$\delta g^{\mu\nu} = g^{\mu\rho}\partial_\rho\varepsilon^\nu + g^{\nu\rho}\partial_\rho\varepsilon^\mu - \varepsilon^\rho\partial_\rho g^{\mu\nu}$$

  – Transition to light-cone variables in terms of $\varepsilon^\pm = \varepsilon^1 - \varepsilon^0 h^\pm$

$$\delta h^\pm = \partial_0\varepsilon^\pm + h^\pm\partial_1\varepsilon^\pm - \varepsilon^\pm\partial_1 h^\pm$$

$$\delta F = -\partial_1(\varepsilon^+ + \varepsilon^-) + (\varepsilon^- - \varepsilon^+)\frac{\partial_1(h^- + h^+)}{h^- - h^+}$$
$$-\frac{\varepsilon^+}{h^- - h^+}(\partial_0 F + h^-\partial_1 F) + \frac{\varepsilon^-}{h^- - h^+}(\partial_0 F + h^+\partial_1 F)$$

  – Closure of algebra

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]h^\pm = \delta(\varepsilon_3)h^\pm \quad [\delta(\varepsilon_1), \delta(\varepsilon_2)]F = \delta(\varepsilon_3)F$$

  – Structure functions

$$\varepsilon_3^\pm = \varepsilon_1^\pm\partial_1\varepsilon_2^\pm - \varepsilon_2^\pm\partial_1\varepsilon_1^\pm$$

Classical algebra

- Generators

$$\varepsilon^{\pm} \to T_{\pm} \qquad \varepsilon \odot T \equiv \int d\sigma[\varepsilon^{+}(\sigma)T_{+}(\sigma)+\varepsilon^{-}(\sigma)T_{-}(\sigma)]$$

Algebra

$$\{\varepsilon_1 \odot T, \varepsilon_2 \odot T\} = \varepsilon_3 \odot T$$

- – From particular expression for structure functions

$$\{T_{\pm}(\sigma), T_{\pm}(\bar{\sigma})\} = [T_{\pm}(\sigma) + T_{\pm}(\bar{\sigma})]\delta'$$

$$\{T_{\pm}, T_{\mp}\} = 0$$

  – Note that it is Virasoro algebra with
  $\varepsilon^{\pm} = \varepsilon^{\pm}(\xi^{+}, \xi^{-})$.
  – Algebra of 2D diffeomorphisms:
  two independent copies of Virasoro algebra
  – Replace conformal invariance prescription by
  principle of 2D reparametrization invariance
  – 2d metric (conformal part F) is quantized

Quantum algebra

- – Transition from classical to quantum theory
  Fields → Operators

$$x^\mu \to \hat{x}^\mu, \qquad \pi_\mu \to \hat{\pi}_\mu, \qquad T_\pm(\varphi) \to : \hat{T}_\pm(\varphi) :$$

  $\varphi = \{G_{\mu\nu},\, B_{\mu\nu},\, \Phi\}$
  – Poisson brackets → Commutators

$$\{A, B\} = C \to [\hat{A}, \hat{B}] = i\hbar\hat{C}$$

- – Breaking of Virasoro algebra (Breaking of 2D diffeomorphisms) because of normal ordering

$$[\hat{T}_\pm(\sigma), \hat{T}_\pm(\bar{\sigma})] = i\hbar[\hat{T}_\pm(\sigma)+\hat{T}_\pm(\bar{\sigma})]\delta'+(\beta^G_{\mu\nu}+\beta^B_{\mu\nu})\hat{O}^{\mu\nu}\delta''+\beta^\Phi\delta'''$$

$$[\hat{T}_\pm(\sigma), \hat{T}_\mp(\bar{\sigma})] = 0$$

  – Quantum 2D diffeomorphisms $\Longleftrightarrow$
  Classical space-time equations of motions

$$\beta^G_{\mu\nu}(\varphi) = 0, \quad \beta^B_{\mu\nu}(\varphi) = 0, \quad \beta^\Phi(\varphi) = 0$$

- – It is possible to determine **exact solutions** and **exact symmetries** of the space-time field equations, which are **not known in an explicit form**
  – **Exact solutions**: String propagates in group manifold
  $G_{\mu\nu} \to$ group metric
  $B_{\mu\nu\rho} \to$ parallelizing torsion
  $\Phi = const$

Exact symmetries

- – Generalization of approach by
  M. Evans, B. A. Ovrut, I. Giannakis, ...
  based on conformal field theory
  – Replace conformal invariance prescription by
  principle of 2D reparametrization invariance
  – 2d metric (conformal part F) is quantized

- – Fields variations $\varphi \to \varphi + \delta\varphi$
  $\hat{T}_\pm(\varphi) \to \hat{T}_\pm(\varphi + \delta\varphi) = \hat{T}_\pm(\varphi) + \delta\hat{T}_\pm(\varphi)$
  – If $\varphi + \delta\varphi$ is also equation of motion

$$[\hat{T}_\pm(\varphi+\delta\varphi)_\sigma, \hat{T}_\pm(\varphi+\delta\varphi)_{\bar{\sigma}}] = i\hbar[\hat{T}_\pm(\varphi+\delta\varphi)_\sigma+\hat{T}_\pm(\varphi+\delta\varphi)_{\bar{\sigma}}]\delta'$$

$$[\hat{T}_\pm(\varphi + \delta\varphi)_\sigma, \hat{T}_\mp(\varphi + \delta\varphi)_{\bar{\sigma}}] = 0$$

  – Conditions

$$[\hat{T}_\pm(\sigma), \delta\hat{T}_\pm(\bar{\sigma})]+[\delta\hat{T}_\pm(\sigma), \hat{T}_\pm(\bar{\sigma})] = i\hbar[\delta\hat{T}_\pm(\sigma)+\delta\hat{T}_\pm(\bar{\sigma})]\delta'$$

$$[\hat{T}_\pm(\sigma), \delta\hat{T}_\mp(\bar{\sigma})] + [\delta\hat{T}_\pm(\sigma), \hat{T}_\mp(\bar{\sigma})] = 0$$

- General solution

$$\delta\hat{T}_\pm(\sigma) = [\hat{\Gamma}_\Lambda, \hat{T}_\pm(\sigma)] \qquad \hat{\Gamma}_\Lambda = \int d\sigma \Upsilon(\Lambda(x), \hat{O})$$

Example

- – Energy-momentum tensor quadratic in currents

$$T_{\pm} = \mp \frac{1}{4\kappa} G^{\mu\nu} j_{\pm\mu} j_{\pm\nu} \qquad j_{\pm\mu} = \pi_\mu + 2\kappa (B_{\mu\nu} \pm \tfrac{1}{2} G_{\mu\nu}) x^{\nu\prime}$$

$$\delta T_{\pm} = \frac{1}{2\kappa} (\delta B_{\mu\nu} \pm \tfrac{1}{2} \delta G_{\mu\nu}) j_{\pm}^{\mu} j_{\pm}^{\nu}$$

  – Chose the form of generator

$$\Gamma_\Lambda = 2\kappa \int d\sigma \Lambda_\mu x^{\mu\prime}$$

$$[\Gamma_\Lambda, T_{\pm}(\sigma)] = \frac{1}{2\kappa} (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu) j_{\pm}^{\mu} j_{\pm}^{\nu}$$

  – Symmetry transformations

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \qquad \delta G_{\mu\nu} = 0$$

- We are going to find symmetries of
  – Open string theory
  – Theory in presence of dilaton field