Properties of Gauged Sigma Models

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24 March 2007 Nis, Serbia

Overview

- Motivation
- Global model
- Lifted formulation
- Gauging
- G/H models
- Example from 6D
- SUGR solution with active G/H hyperscalars

A nice review by E. Sezgin and R. Percacci: Properties of gauged sigma models, hep-th/9810183

- Main Motivation: They arise naturally in globally or locally supersymmetric field theories.

- Scalar fields which parametrize the sigma model manifold either arise from matter or supergravity multiplets

- It is important to understand the structure of supergravity theories in the presence of scalar fields: Dualities, moduli problem etc.

An uncomplete list of sigma model manifolds that arise in SUGR:

D	N	Scalar Manifold G/H	Gauge Group $K \subseteq G$	Matter Sector
10	(2,0)	$SU(1,1)/U(1)$	—-	—-
9	2	$GL(2,R)/SO(2)$	$SO(2)$	—-
	1	$SO(n,1)/SO(n)$	dim K $\subseteq$ n + 1	n Maxwell
8	2	$SL(3,R)/SO(3) \times SL(2,R)/SO(2)$	$SO(3)$	—-
	1	$SO(n,2)/SO(n) \times SO(2)$	dim K $\subseteq$ n + 2	n Maxwell
7	2	$SL(5,R)/SO(5)$	$SO(5)$	—-
	1	$SO(n,3)/SO(n) \times SO(3)$	dim K $\subseteq$ n + 3	n Maxwell
6	(2,2)	$SO(5,5)/SO(5) \times SO(5)$	$SO(5)$	—-
	(2,0)	$SO(n,5)/SO(n) \times SO(5)$	—	n Tensor
	(1,1)	$SO(n,4)/SO(n) \times SO(4)$	dim K $\subseteq$ n + 4	n Maxwell
	(1,0)	Quaternionic Kahler	$Sp(1) \times K'$	n Hyper
		$SO(n,1)/SO(n)$	—-	n Tensor
5	4	$E_6/USp(8)$	$SO(6)$	—-
	3	$SU^*(6)/USp(6)$	$SU(3) \times U(1)$	—-
	2	$SO(n,5)/SO(n) \times SO(5)$	dim K $\subseteq$ n + 5	n Maxwell
		Quaternionic Kahler	$Sp(1) \times K'$	n Hyper
	1	$SO(n-1,1) \times SO(1,1)/SO(n-1)$	dim K $\subseteq$ n	n Maxwell
		$E_{6(-26)}/F_4$	$SU(3)$	25 Maxwell
		$SU^*(6)/Sp(3)$	$SU(3)$	13 Maxwell
		$SL(3,C)/SU(3)$	$SU(3)$	7 Maxwell
		$SL(3,R)/SO(3)$	$SO(2)$	4 Maxwell

Table 1: Supergravities in $D > 4$ dimensions with N supersymmetry and nontrivial sigma model sectors.

The global model:

$$\mathcal{L}_\varphi = -\frac{1}{2f^2}\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu\varphi^\alpha\partial_\nu\varphi^\beta\ g_{\alpha\beta}(\varphi) \qquad\qquad \varphi : M \to N\ .$$

There is a global symmetry group G acting on N.

$$[T_I, T_J] = f_{IJ}{}^K T_K\ ; \qquad\qquad Tr(T_I T_J) \stackrel{-}{=} -\tfrac{1}{2}\delta_{IJ}.$$

The “left” action is generated by vector fields obeying:

$$\mathcal{L}(G) : \qquad \mathcal{L}_{K_I} K^\alpha_J = -f_{IJ}{}^L K_L{}^\alpha\ .$$

The “infinitesimal” global transformation:

$$\delta_\Lambda\varphi^\alpha = -\Lambda^I K^\alpha_I(\varphi)\ , \qquad \partial_\mu\Lambda^I = 0\ .$$

For invariance of the action: $\mathcal{L}_{K_I} g_{\alpha\beta} = 0$.

Kı should be Killing vectors.

The Lifted Formulation: necessary for fermion couplings

Imagine the existence of a larger space: $\bar{N}$

Projection: $\pi : \bar{N} \to N$,

We will assume that π amounts to factoring out a right action of some group H.

$$\mathcal{L}(H) : \qquad \mathcal{L}_{F_a} F_b{}^\gamma = f_{ab}{}^c F_c{}^\gamma \ .$$

$$\varphi : M \to N, \qquad \bar{\varphi} : M \to \bar{N} \ , \qquad \pi\big(\bar{\varphi}(x)\big) \ = \ \varphi(x).$$

The lift is not unique: $\bar{\varphi}'(x) = \big(\bar{\varphi}(x)\big)h(x)$

some map $h : M \to H$,

φ' is also a lift of φ

⇒ there exists a gauge symmetry

$$\delta_\eta \bar{\varphi}^{\bar{\alpha}} = \eta^a F_a^{\bar{\alpha}}(\bar{\varphi})$$

On $\bar{N}$ there must also be the global G invariance:

$$\delta_\Lambda \bar{\varphi}^{\bar{\alpha}} = -\Lambda^I \bar{K}_I^{\bar{\alpha}}(\bar{\varphi}) \ , \qquad \partial_\mu \Lambda^I = 0 \ .$$

We must have: $T\pi(\bar{K}_I) = K_I$

which implies: $\mathcal{L}_{F_a} \bar{K}_I^{\bar{\beta}} = 0$.

The final ingredient is a connection on the bundle

$$\pi : \bar{N} \to N .$$

Vertical subspaces are determined by the kernel of the map π.

To define horizontal subspaces we need Lie algbra valued 1-form : Kernel≡horizontal

$$\omega^a_{\bar{\alpha}} F_b{}^{\bar{\alpha}} = \delta^a{}_b \ .$$

H connection: $$\mathcal{L}_{F_a} \omega^b_{\bar{\alpha}} = -f_{ac}{}^b \omega^c_{\bar{\alpha}} \ .$$

G invariant: $$\mathcal{L}_{\bar{K}_I} \omega^b_{\bar{\alpha}} = 0 \ .$$

The vertical and horizontal projections:

$$V^{\bar\alpha}{}_{\bar\beta} = F_a{}^{\bar\alpha}\omega^a{}_{\bar\beta}\ ,$$
$$H^{\bar\alpha}{}_{\bar\beta} = \delta^{\bar\alpha}{}_{\bar\beta} - V^{\bar\alpha}{}_{\bar\beta}$$

Thus one can define a covariant derivative:

$$\begin{aligned} D_\mu\bar\varphi^{\bar\alpha} &= H^{\bar\alpha}{}_{\bar\beta}\partial_\mu\bar\varphi^{\bar\beta} \\ &= \partial_\mu\bar\varphi^{\bar\alpha} - B^a_\mu F_a{}^{\bar\alpha}(\bar\varphi)\ , \end{aligned}$$

where

$B^a_\mu = \partial_\mu\bar\varphi^{\bar\beta}\omega^a_{\bar\beta}(\varphi)$ is the composite gauge field.

This composite gauge field transforms as a genuine gauge field:

$$\delta_\Lambda B^a_\mu = 0 \ ,$$
$$\delta_\eta B^a_\mu = \partial_\mu B^a_\eta + f^a{}_{bc} B^b_\mu \eta^c \ .$$

Covariant derivatives have the desired transformation properties:

$$\delta_\Lambda D_\mu \bar\varphi^{\bar\alpha} = -\Lambda^I \partial_{\bar\beta} \bar K^{\bar\alpha}_I D_\mu \bar\varphi^{\bar\beta} \ ,$$
$$\delta_\eta D_\mu \bar\varphi^{\bar\alpha} = \eta^a \partial_{\bar\beta} F^{\bar\alpha}_a D_\mu \bar\varphi^{\bar\beta} \ .$$

The lift of the metric: $\bar{g} = \bar{g}_{\bar{\alpha}\bar{\beta}}\, d\bar{y}^{\bar{\alpha}} \otimes d\bar{y}^{\bar{\beta}}$

Properties: V⊥H, should agree with the metric on N

$$\mathcal{L}_{\bar{\varphi}} = -\frac{1}{2}\,\bar{g}_{\bar{\alpha}\bar{\beta}}(\bar{\varphi}) D^{\mu}\bar{\varphi}^{\bar{\alpha}} D_{\mu}\bar{\varphi}^{\bar{\beta}} \ .$$

Because of the gauge invariance, this is equivalent to the original lagrangian.

The gauging of a subgroup K of G:

$$T_i \; (i = 1, ..., \dim K). \qquad A_\mu = A_\mu^i T_i.$$

$$\delta_\Lambda \bar{\varphi}^{\bar{\alpha}} = -\Lambda^i(x) \bar{K}_i^{\bar{\alpha}} \; , \quad \leftarrow \text{local}$$

$$\delta_\Lambda A_\mu^i = \partial_\mu \Lambda^i + g f^i{}_{jk} A_\mu^j \Lambda^k \; , \qquad \delta_\eta A_\mu^i = 0 \; .$$

Introduce the covariant derivative:

$$\mathcal{D}_\mu \bar{\varphi}^{\bar{\alpha}} = \nabla_\mu \bar{\varphi}^{\bar{\alpha}} - B_\mu^a F_a^{\bar{\alpha}}(\bar{\varphi}) \; ,$$

where

$$\nabla_\mu \bar{\varphi}^{\bar{\alpha}} = \partial_\mu \bar{\varphi}^{\bar{\alpha}} + A_\mu^i \bar{K}_i^{\bar{\alpha}}(\bar{\varphi}) \; .$$

The desired transformation properties:

$$\delta_\Lambda \mathcal{D}_\mu \bar{\varphi}^{\bar{\alpha}} = -\Lambda^i(x)\partial_{\bar{\beta}} \bar{K}_I^{\bar{\alpha}} \mathcal{D}_\mu \bar{\varphi}^{\bar{\beta}} \ ,$$

$$\delta_\eta \mathcal{D}_\mu \bar{\varphi}^{\bar{\alpha}} = \eta^a \partial_{\bar{\beta}} F_a^{\bar{\alpha}} \mathcal{D}_\mu \bar{\varphi}^{\bar{\beta}} \ ,$$

gauged sigma model action:

$$\mathcal{L} = -\frac{1}{2}\, \bar{g}_{\bar{\alpha}\bar{\beta}}(\bar{\varphi}) \mathcal{D}^\mu \bar{\varphi}^{\bar{\alpha}} \mathcal{D}_\mu \bar{\varphi}^{\bar{\beta}} \ .$$

The gauge invariant potential:

- No unique way to fix in bosonic theories
- Noether procedure (in general) uniquely fixes the potential in supersymmetric theories.

Possible to introduce a Wess-Zumino term

G/H models:

N : G/H

one can apply the lifted formulation!

$\bar{N}$: G

But there is a gauged fixed version which is very practical

Coset structure in the algebra: G=H⊕P

[H,H]∈H $\{T_a\}$ $a = 1, \ldots, dim\, H$

[P,P]∈H $\{T_r\}$ $r = 1, \ldots, dim\, G/H$

[H,P]∈P

$L(y)$: coset representative: G-valued

The action: G $L(y)$ H

Construct Maurer Cartan form: $L^{-1}\partial_\alpha L = V_\alpha^r T_r + B_\alpha^a T_a$,

V_α^r : vielbein

B_α^a : gauge potential

Spacetime pull backs:

$$L^{-1}\partial_\mu L = P_\mu^r T_r + B_\mu^a T_a \ ,$$

where $P_\mu^r = \partial_\mu \varphi^\alpha V_\alpha^r \ , \qquad B_\mu^a = \partial_\mu \varphi^\alpha B_\alpha^a \ .$

ungauged sigma model $\mathcal{L}_0 = \frac{1}{2} P_{\mu r} P^{\mu r}$.

For gauging introduce the covariant derivative:

$$L^{-1}\left(\partial_\mu + A_\mu^i T_i\right) L = \mathcal{P}_\mu^r T_r + \mathcal{B}_\mu^a T_a \ .$$

The potential becomes a function of the so called C-functions:

$$L^{-1} T^I L \equiv C^I$$

V= $\operatorname{tr} C_i C^i$.

The difficult part is to find a convenient parametrization of the coset.

Try to introduce coordinates covering the whole manifold.

Example:

from a recent work with E. Sezgin and D. Jong

6D dyonic string with active hyperscalars,
hep-th: 0608034

6D N=(1,0) gauged supergravity coupled to tensor and vector multiplets.

Motivation: find the structure of the most general supersymmetric solutions with active hyperscalars parametrizing a coset space.

The coset space: $Sp(n_H, 1)/Sp(n_H) \times Sp(1)$

Sp(n) can naturally be defined using quaternions.

n × n Hermitian matrix of quaternions

non-compact Sp(n,1) can be defined similarly.

For the compact case the parametrization of coset was given by Gursey and Tze.

Non-compact generalization by Sezgin.

$$L = \gamma^{-1} \begin{pmatrix} 1 & t^\dagger \\ t & \Lambda \end{pmatrix} \qquad t^p \; (p = 1, ..., n_H),$$

$$\gamma = (1 - t^\dagger\, t)^{1/2} \;, \qquad \Lambda = \gamma\, (I - t\, t^\dagger)^{-1/2} \;.$$

The gauged Maurer-Cartan form:

$$L^{-1} D_\mu L = \begin{pmatrix} Q_\mu & P_\mu^\dagger \\ P_\mu & Q'_\mu \end{pmatrix}$$

$$
\begin{aligned}
Q_\mu &= \frac{1}{2}\gamma^{-2}\left(D_\mu t^\dagger t - t^\dagger D_\mu t\right) - A^r_\mu T^r \\
Q'_\mu &= \gamma^{-2}\left(-tD_\mu t^\dagger + \Lambda D_\mu \Lambda + \tfrac{1}{2}\partial_\mu (t^\dagger t) I\right) - A^{I'}_\mu T^{I'} \ , \\
P_\mu &= \gamma^{-2}\Lambda D_\mu t \ ,
\end{aligned}
$$

where

$$
D_\mu t = \partial_\mu t + t\, T^r A^r_\mu - A^{I'}_\mu T^{I'}\, t \ .
$$

C-functions:

$$
C^r = L^{-1} T^r L = \gamma^{-2}\begin{pmatrix} T^r & T^r t^\dagger \\ -tT^r & -tT^r t^\dagger \end{pmatrix}
$$

$$
C^{I'} = L^{-1} T^{I'} L = \gamma^{-2}\begin{pmatrix} -t^\dagger T^{I'} t & -t^\dagger T^{I'}\Lambda \\ \Lambda T^{I'} t & \Lambda T^{I'}\Lambda \end{pmatrix}
$$

For the coset space Sp(1,1)/Sp(1)xSp(1)

$$V = \frac{4}{(1-\phi^2)^2} \left[g_R^2 + g'^2 (\phi^2)^2 \right]$$

φ is a vector parametrizing 4-dim. hyperboloid.

For higher dimensional cases, the potential is much more complicated.

However, it has a stable global minimum at t=0.

The solution with active hyperscalars:

The D=6 model:

$$\mathcal{L} = R - \tfrac{1}{4}(\partial\varphi)^2 - \tfrac{1}{12}e^{\varphi}\, G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{4}\, e^{\frac{1}{2}\varphi}\, F^{I}_{\mu\nu}\, F^{I\,\mu\nu} - 2P^{aA}_{\mu}\, P^{\mu}_{aA} - 4\, e^{-\frac{1}{2}\varphi}\, C^{I}_{AB}C^{I\,AB}\ ,$$

$$\begin{aligned}
\delta\psi_\mu &= D_\mu\varepsilon + \tfrac{1}{48}e^{\frac{1}{2}\varphi}G^{+}_{\nu\sigma\rho}\,\Gamma^{\nu\sigma\rho}\,\Gamma_\mu\,\varepsilon\ ,\\
\delta\chi &= \tfrac{1}{4}\left(\Gamma^{\mu}\partial_\mu\varphi - \tfrac{1}{6}e^{\frac{1}{2}\varphi}G^{-}_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\right)\varepsilon\ ,\\
\delta\lambda^{I}_{A} &= -\tfrac{1}{8}F^{I}_{\mu\nu}\Gamma^{\mu\nu}\varepsilon_A - e^{-\frac{1}{2}\varphi}C^{I}_{AB}\,\varepsilon^{B}\ ,\\
\delta\psi^{a} &= P^{aA}_{\mu}\Gamma^{\mu}\varepsilon_A\ ,
\end{aligned}$$

Higher dimensional origin is not known!!

Conditions from the existence of a Killing spinor:

- There exists a null Killing vector: V_μ
- There exists a quaternionic structure obeying

$$(I^r)^i{}_k\,(I^s)^k{}_j = \epsilon^{rst}(I^t)^i{}_j - \delta^{rs}\delta^i_j \ .$$

Conditions from the hyperfermion variation:

$$\begin{aligned} V^\mu P^{aA}_\mu &= 0 \ , \\ P^{aA}_i &= 2(I^r)_i{}^j\,(T^r)^A{}_B\,P^{aB}_j \ . \end{aligned}$$

First order equation for scalars, similar to a holomorphicity condition.

Ansatz:

$$t = \begin{pmatrix} \phi \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$D_i \phi^i = 0 \ , \qquad \phi^i \equiv \phi^{\underline{\alpha}} \delta^i_{\underline{\alpha}} \ ,$$

$$D_i \phi_j - D_j \phi_i = -\epsilon_{ijk\ell} D_k \phi_\ell \ .$$

2+4 split of the geometry, identity map:

$$
\begin{aligned}
ds^2 &= e^{-\frac{1}{2}\varphi_+}e^{-\frac{1}{2}\varphi_-}(-dt^2+dx^2)+L^2e^{\frac{1}{2}\varphi_+}e^{\frac{1}{2}\varphi_-}\,h^{2/3}\,(dr^2+r^2d\Omega_3^2)\\
e^{\varphi} &= e^{\varphi_+}/e^{\varphi_-}\ ,\\
G &= \tfrac{8}{27}\,\Omega_3 - dt\wedge dx\wedge de^{-\varphi_+}\ ,\\
A^r &= \tfrac{2}{3}\,r^2\sigma_R^r\ ,\\
\phi^\alpha &= z^\alpha\ ,
\end{aligned}
$$

where

$$
r=\sqrt{z^\alpha z^\beta\delta_{\alpha\beta}}\ ,\qquad \Omega_3=\sigma_R^1\wedge\sigma_R^2\wedge\sigma_R^3\ ,\qquad h=\frac{1}{r^2}-1\ ,
$$

$$
e^{\varphi_+}=\frac{3\nu h^{1/3}}{L^2}+\nu_0\ ,\qquad e^{\varphi_-}=\frac{4h^{1/3}}{9L^2}\ ,
$$

Dyonic string solution, 1/8 susy, tear drop, non-compact but finite volume transverse space singular!!, for some parameters AdS3xS3 horizon