Quantum fields on the de Sitter spacetime

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Abstract

The theory of external symmetry in curved spacetimes we have proposed few years ago allows us to correctly define the operators of the quantum field theory on curved backgrounds. Particularly, despite of some doubts appeared in literature, we have shown that a well-defined energy operator can be considered on the de Sitter manifold. With its help new quantum modes were obtained for the scalar, Dirac and vector fields on the de Sitter spacetimes. A short review of these results is presented in this report.

Symmetries and conserved observables

The quantum field theory on curved manifolds must be build in ***local*** (unholonomic) frames where the spin half can be well-defined.

The relativistic covariance and the gauge symmetry

Let us consider the manifold (M, g) and a local chart (or natural frame) $\{x\}$ of coordinates x^{μ} ($\mu, \nu, .. = 0, 1, 2, 3$). The general relativistic covariance preserves the form of the field equations under any coordinate transformation $x \to x' = \phi(x)$.

The local frames are given by the tetrad fields $e_{\hat{\mu}}(x)$ and $\hat{e}^{\hat{\mu}}(x)$, labeled by the local indices $\hat{\alpha}, \hat{\beta}, ... = 0, 1, 2, 3$, which have the usual duality and orthonormalization properties

$$\hat{e}^{\hat{\mu}}_{\alpha}\, e^{\alpha}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}\,, \quad \hat{e}^{\hat{\mu}}_{\alpha}\, e^{\beta}_{\hat{\mu}} = \delta^{\beta}_{\alpha}\,, \quad e_{\hat{\mu}} \cdot e_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}\,, \quad \hat{e}^{\hat{\mu}} \cdot \hat{e}^{\hat{\nu}} = \eta^{\hat{\mu}\hat{\nu}}\,, \tag{1}$$

where $\eta =$diag$(1, -1, -1, -1)$ is the Minkowski metric. This metric remains invariant under the transformations $\Lambda[A(\omega)]$ of the gauge group $G(\eta) = L^{\uparrow}_{+}$ corresponding to the transformations $A(\omega) \in SL(2,\mathbb{C})$ through the canonical homomorphism. In the standard parametrization, with $\omega^{\hat{\alpha}\hat{\beta}} = -\omega^{\hat{\beta}\hat{\alpha}}$, we have

$$A(\omega) = e^{-\frac{i}{2}\omega^{\hat{\alpha}\hat{\beta}}S_{\hat{\alpha}\hat{\beta}}}\,, \tag{2}$$

where $S_{\hat{\alpha}\hat{\beta}}$ are the covariant basis-generators of the $sl(2,\mathbb{C})$ algebra. For small values of $\omega^{\hat{\alpha}\hat{\beta}}$ the matrix elements of Λ can be written as

$$\Lambda[A(\omega)]^{\hat{\mu}\,\cdot}_{\cdot\,\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}} + \omega^{\hat{\mu}\,\cdot}_{\cdot\,\hat{\nu}} + \cdots\,. \tag{3}$$

The theory contains matter fields ψ transforming according to a representation ρ of the $SL(2,\mathbb{C})$ group. The entire theory must be ***gauge invariant*** in the sense that it remains invariant when one performs a gauge transformation

$$\psi(x) \;\rightarrow\; \psi'(x) = \rho[A(x)]\psi(x) \tag{4}$$

$$e_{\hat{\alpha}}(x) \;\rightarrow\; e'_{\hat{\alpha}}(x) = \Lambda^{\cdot\,\hat{\beta}}_{\hat{\alpha}\,\cdot}[A(x)]e_{\hat{\beta}}(x) \tag{5}$$

produced by the point-dependent transformations $A(x) \in SL(2,\mathbb{C})$ and $\Lambda[A(x)] \in L^{\uparrow}_{+}$.

The general relativistic covariance as well as the tetrad-gauge invariance are not able to give rise to conserved quantities. These are produced by isometries.

The external symmetry

In general, (M, g) can have isometries that form the isometry group $I(M)$ whose parameters are denotad by ξ_a ($a, b, ... = 1..N$). Given an isometry $\phi_\xi \in I(M)$ then, for each parameter ξ_a, there exists an associated Killing vector field defined as

$$K_a = \partial_{\xi_a}\phi_\xi|_{\xi=0}. \tag{6}$$

Starting with an isometry $x \to x' = \phi_\xi(x)$ we introduced the so called external symmetry transformations, (A_ξ, ϕ_ξ), defined as ***combined*** transformations involving gauge transformations necessary to preserve the gauge [1],

$$\Lambda[A_\xi(x)]^{\hat{\alpha}\,\cdot}_{\cdot\,\hat{\beta}} = \hat{e}^{\hat{\alpha}}_{\mu}[\phi_\xi(x)]\frac{\partial\phi^{\mu}_{\xi}(x)}{\partial x^\nu}\, e^{\nu}_{\hat{\beta}}(x)\,, \tag{7}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2, \mathbb{C})$. The transformations (A_ξ, ϕ_ξ) leave the field equation invariant and constitute the ***group of external symmetry***, $S(M)$, which is the universal covering group of $I(M)$.

The transformations of the group $S(M)$ are

$$(A_\xi, \phi_\xi): \qquad \begin{array}{rcrcl} x & \to & x' & = & \phi_\xi(x) \\ e(x) & \to & e'(x') & = & e[\phi_\xi(x)] \\ \hat{e}(x) & \to & \hat{e}'(x') & = & \hat{e}[\phi_\xi(x)] \\ \psi(x) & \to & \psi'(x') & = & \rho[A_\xi(x)]\psi(x)\,. \end{array} \tag{8}$$

In [11] we presented arguments that $S(M)$ is the universal covering group of $I(M)$.

For small ξ^a the covariant $SL(2,\mathbb{C})$ parameters of $A_\xi(x) \equiv A[\omega_\xi(x)]$ can be expanded as $\omega_\xi^{\hat{\alpha}\hat{\beta}}(x) = \xi^a \Omega_a^{\hat{\alpha}\hat{\beta}}(x) + \cdots$ where the functions

$$\Omega_a^{\hat{\alpha}\hat{\beta}} \equiv \frac{\partial \omega_\xi^{\hat{\alpha}\hat{\beta}}}{\partial \xi^a}_{|\xi=0} = \left(\hat{e}_\mu^{\hat{\alpha}}\, K_{a,\nu}^\mu + \hat{e}_{\nu,\mu}^{\hat{\alpha}} K_a^\mu\right) e_{\hat{\lambda}}^\nu \eta^{\hat{\lambda}\hat{\beta}}\,. \tag{9}$$

depend on the Killing vectors (6).

The generators of any representation

The matter field ψ transforms according to the operator-valued representation $(A_\xi, \phi_\xi) \rightarrow T^\rho_\xi$ which is defined as

$$(T^\rho_\xi \psi)[\phi_\xi(x)] = \rho[A_\xi(x)]\psi(x) \qquad (10)$$

and leaves *invariant* the field equation in local frames, $T^\rho_\xi E (T^\rho_\xi)^{-1} = E$.

The basis-generators of the representations ρ of the $s(M)$ algebra,

$$X^\rho_a = i \frac{\partial T^\rho_\xi}{\partial \xi^a}_{|\xi=0} = L_a + S^\rho_a \,, \qquad (11)$$

are formed by orbital parts, $L_a = -iK^\mu_a \partial_\mu$, and spin terms [1],

$$S^\rho_a(x) = i \frac{\partial A_\xi(x)}{\partial \xi^a}_{|\xi=0} = \frac{1}{2}\, \Omega^{\hat\alpha\hat\beta}_a(x) \rho(S_{\hat\alpha\hat\beta}) \,, \qquad (12)$$

that depend on the functions (9). These operators can be written in ***covariant*** form according to the Carter and McLenaghan formula [2]

$$X_a^\rho = -iK_a^\mu D_\mu^\rho + \frac{1}{2}\, K_{a\,\mu;\nu}\, e^\mu_{\hat\alpha}\, e^\nu_{\hat\beta}\, \rho(S^{\hat\alpha\hat\beta})\,. \tag{13}$$

where D_μ^ρ are the covariant derivatives in local frames associated to the representation ρ of $SL(2,\mathbb{C})$.

However, whenever the mater fields are vector and tensors, then the basis-generators of a tensor representation ρ_n of the rank n can be written in natural frames,

$$X_a^n = -iK_a^\mu \nabla_\mu + \frac{1}{2}\, K_{a\,\mu;\,\nu}\, \rho_n(\tilde S^{\mu\nu})\,, \tag{14}$$

where ∇_μ are the usual covariant derivatives. The point-dependent generators $\tilde S(x)$ are defined as

$$(\tilde S^{\mu\nu})^{\sigma\,\cdot}_{\cdot\,\tau} = e^\mu_{\hat\alpha} e^\nu_{\hat\beta} e^\sigma_{\hat\gamma} [\rho_v(S^{\hat\alpha\hat\beta})]^{\hat\gamma\,\cdot}_{\cdot\,\hat\delta} \hat e^{\hat\delta}_\tau = i(g^{\mu\sigma}\delta^\nu_\tau - g^{\nu\sigma}\delta^\mu_\tau)\,. \tag{15}$$

We say that the operators (25) and (26) are ***conserved*** operators since they commute with the operators of the field equations.

The de Sitter spacetime

Let (M, g) be the de Sitter (dS) spacetime defined as a hyperboloid in the $(1+4)$ - dimensional flat spacetime, $({}_5M, {}_5\eta)$, of coordinates z^A, $A, B, ... = 0, 1, 2, 3, 5$, and metric ${}_5\eta = \mathrm{diag}(1, -1, -1, -1, -1)$,

$$ {}_5\eta_{AB} z^A z^B = -R^2 , \qquad R = 1/\omega = \sqrt{3/|\Lambda_c|} \, . \tag{16} $$

The gauge group of the metric ${}_5\eta$ plays the role of ***isometry*** group of the dS manifold, $G[{}_5\eta] = I(M) = SO(1,4)$. We use covariant real parameters, ${}_5\omega^{AB} = -{}_5\omega^{BA}$, since in this parametrization the orbital basis-generators of the scalar representation of $G({}_5\eta)$, carried by the spaces of functions over ${}_5M$, has the usual form

$$ {}_5L_{AB} = i \left[{}_5\eta_{AC} \, z^C \partial_B - \, {}_5\eta_{BC} \, z^C \partial_A \right] . \tag{17} $$

A local chart $\{x\}$ on (M, g), of coordinates x^μ, $\mu, \nu, ... = 0, 1, 2, 3$, is defined by the functions $z^A = z^A(x)$. The identification ${}_5L_{AB} = -iK^\mu_{(AB)}\partial_\mu$ defines the components in the chart $\{x\}$ of the Killing vector field $K_{(AB)}$ associated to ${}_5\omega^{AB}$.

Static charts

with conformal spherical line element:
1. Cartesian coordinates: $\{t_s, \vec{x}_s\}$
2. spherical coordinates: $\{t_s, r_s, \theta, \phi\}$

$$ds^2 = \frac{1}{\cosh^2 \omega r_s} \left[dt_s^2 - dr_s^2 - \frac{1}{\omega^2} \sinh^2 \omega r_s \, (d\theta^2 + \sin^2\theta \, d\phi^2) \right] , \tag{18}$$

with finite event horizon at $\omega \hat{r}_s = 1$:
1. Cartesian coordinates: $\{t_s, \vec{\hat{x}}_s\}$
2. spherical coordinates: $\{t_s, \hat{r}_s, \theta, \phi\}$

$$ds^2 = (1 - \omega^2 \hat{r}_s^2) dt_s^2 - \frac{d\hat{r}_s^2}{1 - \omega^2 \hat{r}_s^2} - \hat{r}_s^2 (d\theta^2 + \sin^2\theta \, d\phi^2) , \tag{19}$$

where

$$\omega \hat{r}_s = \tanh \omega r_s \, . \tag{20}$$

Moving charts

with FRW line element, proper time and:

1. Cartesian coordinates: $\{t, \vec{x}\}$
2. spherical coordinates: $\{t, r, \theta, \phi\}$

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x} \cdot d\vec{x} = dt^2 - e^{2\omega t}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \qquad (21)$$

with conformal flat line element and:

1. Cartesian coordinates: $\{t_c, \vec{x}\}$
2. spherical coordinates: $\{t_c, r, \theta, \phi\}$

$$ds^2 = \frac{1}{\omega^2 t_c^2}(dt_c^2 - d\vec{x} \cdot d\vec{x}) = \frac{1}{\omega^2 t_c^2}[dt_c^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \qquad (22)$$

where $\omega t_c = -e^{-\omega t}$ and:

$$t_s = t - \frac{1}{2\omega}\ln\left(1 - \omega^2 r^2 e^{2\omega t}\right)\,, \quad r_s = \frac{1}{2\omega}\ln\frac{1 + \omega r e^{\omega t}}{1 - \omega r e^{\omega t}}\,, \quad \hat{r}_s = r e^{\omega t}\,. \qquad (23)$$

The principal observables on the de Sitter spacetime

In the case of the dS spacetime, in a chart where we consider the tetrad fields e and $\hat{e}$, we identify $\xi^a \to {}_5\omega^{AB}$ and $a \to (AB)$ such that the generators of an arbitrary representation ρ read

$$X^{\rho}_{(AB)} = L_{(AB)} + S^{\rho}_{(AB)} = -iK^{\mu}_{(AB)}\partial_{\mu} + \frac{1}{2}\,\Omega^{\hat{\alpha}\hat{\beta}}_{(AB)}\rho(S_{\hat{\alpha}\hat{\beta}})\,. \tag{24}$$

The covariant forms can be written in local frames,

$$X^{\rho}_{(AB)} = -iK^{\mu}_{(AB)}D^{\rho}_{\mu} + \frac{1}{2}\,K_{(AB)\,\mu;\nu}\,e^{\mu}_{\hat{\alpha}}\,e^{\nu}_{\hat{\beta}}\,\rho(S^{\hat{\alpha}\hat{\beta}})\,, \tag{25}$$

for any field or even in natural frames but only for the vector and tensor fields,

$$X^{n}_{(AB)} = -iK^{\mu}_{(AB)}\nabla_{\mu} + \frac{1}{2}\,K_{(AB)\,\mu;\,\nu}\,\rho_n(\tilde{S}^{\mu\nu})\,. \tag{26}$$

name	$\{t_s, \vec{x}_s\}$	$\{t, \vec{x}\}$	$\{t_c, \vec{x}\}$
definition	any gauge	diagonal gauge	diagonal gauge
Energy:			
$H = \omega X_{(05)}$	$= i\partial_{t_s}$	$= i\partial_t - i\omega x^i \partial_i$	$-i\omega(t_c\partial_{t_c} + x^i\partial_i)$
Momentum:			
$P_i = \omega(X_{(5i)} - X_{(0i)})$	Ref. [1]	$= -i\partial_i$	id.
Angular momentum:			
$J_{ij} = X_{(ij)}$	sim.	$= -i(x^i\partial_j - x^j\partial_i) + \rho(S_{ij})$	id.

More three generators:

$N_i = \omega(X_{(5i)} + X_{(0i)})$ do not have an immediate physical meaning.

The specific features:

1. The momentum and energy operators do not commute among themselves,

$$[H, P_i] = i\omega P_i\,. \tag{27}$$

2. The energy operator H is well-defined only on restricted domains where $K_{(05)}$ is time-like. For this reason some people asks whether and how the energy can be measured.

The problem of the energy operator

Where the Killing vector $K_{(05)}$ is time-like ?

chart	$\{t_s, \vec{\hat{x}}_s\}$	$\{t_s, \vec{x}_s\}$	$\{t, \vec{x}\}$	$\{t_c, \vec{x}\}$
light-cone domain	$(\omega \hat{r}_s < 1)_{e.h.}$	$r_s < \|t_s\|$	$\omega r e^{\omega t} < 1$	$r < \|t_c\|$
$K_{(05)}$	$(-\frac{1}{\omega}, 0, 0, 0)$	$(-\frac{1}{\omega}, 0, 0, 0)$	$(-\frac{1}{\omega}, x^1, x^2, x^3)$	(t_c, x^1, x^2, x^3)
$g(K_{(05)}, K_{(05)}) > 0$	$1 - \omega^2 \hat{r}_s^2 > 0$	$r_s > 0$	$\frac{1}{\omega^2} - r^2 e^{2\omega t} > 0$	$t_c^2 - r^2 > 0$

Conclusions:

1. The Killing vector $K_{(05)}$ is time-like inside the light-cone of any given chart.

2. The energy operator H is well-defined on the entire domain where an observer can measure physical events.

Quantum modes on the de Sitter manifold

The quantum modes are determined by complete systems of commuting operators (c.s.c.o.) formed by the operator of the field equation E - of the scalar (S), Proca (V), Maxwell (M) and Dirac (D) - and various generators commuting with E. Among them we consider the operators H, P_i, $J_i = \frac{1}{2}\varepsilon_{ijk}J_{jk} = L_i + \rho(S_i)$ and the helicity operator,

$$W = \vec{P}\cdot\vec{J}\,. \tag{28}$$

The operator of the momentum direction, $\vec{\hat{P}}$, is no longer a differential one being defined as

$$P_i = \hat{P}_i\sqrt{\vec{P}^2}\,. \tag{29}$$

In addition we define the normalized helicity operator

$$\hat{W} = \vec{\hat{P}}\cdot\vec{J}\,. \tag{30}$$

These operators obey the following commutation rules:

$$[H, W] = i\omega W\,, \quad [H, \hat{P}_i] = 0\,, \quad [H, \hat{W}] = 0\,. \tag{31}$$

The analytically solvable scalar quantum modes

chart/c.s.c.o.	$\{E_S, H, \vec{L}^2, L_3\}$	$\{E_S, \vec{P}^2, \vec{L}^2, L_3\}$	$\{E_S, P_i\}$	$\{E_S, H, \hat{P}_i\}$
	spherical waves	spherical waves	plane waves	plane waves
$\{t_s, r_s, \theta, \phi\}$.	Avis, Isham, Storey, [3]	***?***	no	no
$\{t, \vec{x}\}$, $\{t_c, \vec{x}\}$	no	no	Chernikov, Tagirov [4]	Cotaescu [5]

The analytically solvable vector quantum modes

chart/c.s.c.o.	$\{E_V, H, \vec{J}^2, J_3\}$	$\{E_V, \vec{P}^2, \vec{J}^2, J_3\}$	$\{E_V, P_i, W\}$	$\{E_V, H, \hat{P}_i, \hat{W}\}$
	spherical waves	spherical waves	plane waves	plane waves
$\{t_s, r_s, \theta, \phi\}$.	Higuchi [6]	***?***	no	no
$\{t, \vec{x}\}$, $\{t_c, \vec{x}\}$	no	no	Cotaescu [7]	($m = 0$) Cotaescu [8]

The analytically solvable Dirac quantum modes

chart/c.s.c.o. gauge	$\{E_D, H, \vec{J}^2, K, J_3\}$ spherical waves	$\{E_D, \vec{P}^2, \vec{J}^2, K, J_3\}$ spherical waves	$\{E_D, P_i, W\}$ plane waves	$\{E_D, H, \hat{P}_i, \hat{W}\}$ plane waves
$\{t_s, r_s, \theta, \phi\}$. diagonal gauge	Otchik [9]	?	no	no
$\{t_s, r_s, \theta, \phi\}$. Cartesian gauge	Cotaescu [10]	?	no	no
$\{t, r, \theta, \phi\}$ Cartesian gauge	Cotaescu [14]	Shishkin [12] Cotaescu [13]	no	no
$\{t, \vec{x}\}$, $\{t_c, \vec{x}\}$ diagonal gauge	no	no	Cotaescu [11]	Cotaescu [14]

where K is the Dirac angular operator.

The free quantum fields in moving charts

In the moving charts, $\{t, \vec{x}\}$ and $\{t_c, \vec{x}\}$, the principal quantum fields minimally coupled to gravity can be expanded as mode integrals in the ***momentum*** basis.

The massive charged scalar field

The scalar field ϕ satisfy the Klein-Gordon equation

$$\left(\partial_t^2 - e^{-2\omega t}\Delta + 3\omega\partial_t + m^2\right)\phi(x) = 0\,. \tag{32}$$

The expansion in the momentum basis,

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) = \int d^3p\left[f_{\vec{p}}(x)a(\vec{p}) + f^*_{\vec{p}}(x)b^*(\vec{p})\right]\,, \tag{33}$$

can be done using the fundamental solutions

$$f_{\vec{p}}(x) = \frac{1}{2}\sqrt{\frac{\pi}{\omega}}\frac{1}{(2\pi)^{3/2}}\,e^{-3\omega t/2}Z_k\left(\frac{p}{\omega}\,e^{-\omega t}\right)e^{i\vec{p}\cdot\vec{x}}\,, \quad Z_k(s) = e^{-\pi k/2}H^{(1)}_{ik}(s) \tag{34}$$

where $k=\sqrt{\frac{m^2}{\omega^2}-\frac{9}{4}}$. These modes satisfy the orthonormalization relations

$$\langle f_{\vec{p}}, f_{\vec{p}'}\rangle = -\langle f^*_{\vec{p}}, f^*_{\vec{p}'}\rangle = \delta^3(\vec{p}-\vec{p}')\,, \quad \langle f_{\vec{p}}, f^*_{\vec{p}'}\rangle = 0\,, \tag{35}$$

with respect to the relativistic scalar product

$$\langle \phi, \phi'\rangle = i\int d^3x\, e^{3\omega t}\, \phi^*(x) \overleftrightarrow{\partial_t}\, \phi'(x)\,, \tag{36}$$

and the completeness condition

$$i\int d^3p\, f^*_{\vec{p}}(t,\vec{x}) \overleftrightarrow{\partial_t}\, f_{\vec{p}}(t,\vec{x}') = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}')\,. \tag{37}$$

The massive charged vector field

This field obeys the Proca-type equation

$$\partial_{t_c}(\partial_i A_i) - \Delta A_0 + \frac{\mu^2}{t_c^2}\, A_0 = 0\,, \tag{38}$$

$$\partial^2_{t_c} A_k - \Delta A_k - \partial_k(\partial_c t A_0) + \partial_k(\partial_i A_i) + \frac{\mu^2}{t_c^2}\, A_k = 0\,, \tag{39}$$

where $\mu = m/\omega$, while the Lorentz condition reads

$$\partial_i A_i = \partial_{t_c} A_0 - \frac{2}{t_c} A_0 \,. \tag{40}$$

The mode expansion in the momentum basis,

$$\begin{aligned} A &= A^{(+)} + A^{(-)} \\ &= \int d^3p \sum_\lambda \left\{ \mathrm{U}[\vec{p},\lambda] a(\vec{p},\lambda) + \mathrm{U}[\vec{p},\lambda]^* b^*(\vec{p},\lambda) \right\} \,, \end{aligned} \tag{41}$$

involve fundamental solutions, $\mathrm{U}[\vec{p},\lambda]$, which satisfy the Proca equation, the Lorenz condition, the eigenvalue equations

$$P^i \mathrm{U}[\vec{p},\lambda] = p_i \mathrm{U}[\vec{p},\lambda] \,, \quad W \mathrm{U}[\vec{p},\lambda] = p\,\lambda \mathrm{U}[\vec{p},\lambda] \,, \tag{42}$$

and the orthonormalization relations

$$\langle \mathrm{U}[\vec{p},\lambda] |\, \mathrm{U}[\vec{p}\,',\lambda'] \rangle = \delta_{\lambda\lambda'} \delta^3(\vec{p} - \vec{p}\,') \,. \tag{43}$$

with respect to the relativistic scalar product

$$\langle A |\, A' \rangle = -\eta^{\mu\nu} \langle A_\mu, A'_\nu \rangle = -i\eta^{\mu\nu} \int d^3x\, A^*_\mu(t_c,\vec{x})\, \overleftrightarrow{\partial}_{t_c} A'_\nu(t_c,\vec{x}) \,. \tag{44}$$

These solutions are of the form

$$\mathrm{U}[\vec{p},\lambda]_i(x) = \begin{cases} \alpha(t_c,p)\, e_i(\vec{n}_p,\lambda)\, e^{i\vec{p}\cdot\vec{x}} & \text{for } \lambda = \pm 1 \\ \beta(t_c,p)\, e_i(\vec{n}_p,\lambda)\, e^{i\vec{p}\cdot\vec{x}} & \text{for } \lambda = 0 \end{cases} \tag{45}$$

and

$$\mathrm{U}[\vec{p},\lambda]_0(x) = \begin{cases} 0 & \text{for } \lambda = \pm 1 \\ \gamma(t_c,p)\, e^{i\vec{p}\cdot\vec{x}} & \text{for } \lambda = 0 \end{cases} \tag{46}$$

where $\vec{n}_p = \vec{p}/p$ and

$$\alpha(t_c,p) \;=\; N_1 e^{-\frac{1}{2}\pi k}(-t_c)^{\frac{1}{2}}\, H^{(1)}_{ik}(-pt_c)\,, \tag{47}$$

$$\gamma(t_c,p) \;=\; N_2 e^{-\frac{1}{2}\pi k}(-t_c)^{\frac{3}{2}}\, H^{(1)}_{ik}(-pt_c)\,, \tag{48}$$

$$\beta(t_c,p) \;=\; iN_2 e^{-\frac{1}{2}\pi k}\left[\frac{1}{p}\left(ik+\frac{1}{2}\right)(-t_c)^{\frac{1}{2}} H^{(1)}_{ik}(-pt_c) - (-t)^{\frac{3}{2}} H^{(1)}_{ik+1}(-pt_c)\right]\,, \tag{49}$$

with the notations $k = \sqrt{\frac{m^2}{\omega^2} - \frac{1}{4}}$ and

$$N_1 = \frac{\sqrt{\pi}}{2}\frac{1}{(2\pi)^{3/2}}\,,\quad N_2 = \frac{\sqrt{\pi}}{2}\frac{1}{(2\pi)^{3/2}}\frac{\omega p}{m}\,. \tag{50}$$

The polarization vectors $\vec{e}(\vec{n}_p, \lambda)$ of the helicity basis are longitudinal for $\lambda = 0$, i.e. $\vec{e}(\vec{n}_p, 0) = \vec{n}_p$, while for $\lambda = \pm 1$ they are transversal, $\vec{p} \cdot \vec{e}(\vec{n}_p, \pm 1) = 0$. They have c-number components which satisfy

$$\vec{e}(\vec{n}_p, \lambda)^* \cdot \vec{e}(\vec{n}_p, \lambda') = \delta_{\lambda\lambda'} \,, \tag{51}$$

$$\vec{e}(\vec{n}_p, \lambda)^* \wedge \vec{e}(\vec{n}_p, \lambda) = i\lambda \, \vec{n}_p \,, \tag{52}$$

$$\sum_{\lambda} e_i(\vec{n}_p, \lambda)^* \, e_j(\vec{n}_p, \lambda) = \delta_{ij} \,. \tag{53}$$

The massless limit makes sense only if we take $\beta(t, p) = \gamma(t, p) = 0$ which leads to the Coulomb gauge of the Maxwell free field, $A_0 = 0$ and $\partial_i A_i = 0$. [8]

The Maxwell field in Coulomb gauge

$$A_0(x) = 0 \,,$$

$$A_i(x) = \int d^3k \sum_{\lambda = \pm 1} \left[e_i(\vec{n}_k, \lambda) f_{\vec{k}}(x) a(\vec{k}, \lambda) + [e_i(\vec{n}_k, \lambda) f_{\vec{k}}(x)]^* a^*(\vec{k}, \lambda) \right] \tag{54}$$

is expanded in terms of fundamental solutions of the d'Alambert equation,

$$f_{\vec{k}}(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}} \, e^{-ikt_c + i\vec{k}\cdot\vec{x}} \,, \tag{55}$$

The massive Dirac field

The Dirac field ψ satisfies the free equation $E_D\psi = m\psi$. In the Cartesian gauge with the non-vanishing tetrad components

$$e_0^0 = -\omega t_c\,, \quad e_j^i = -\delta_j^i\,\omega t_c\,, \quad \hat{e}_0^0 = -\frac{1}{\omega t_c}\,, \quad \hat{e}_j^i = -\delta_j^i\,\frac{1}{\omega t_c}\,, \tag{56}$$

the Dirac operator reads

$$\begin{aligned} E_D &= -i\omega t_c\left(\gamma^0\partial_{t_c} + \gamma^i\partial_i\right) + \frac{3i\omega}{2}\gamma^0 \\ &= i\gamma^0\partial_t + ie^{-\omega t}\gamma^i\partial_i + \frac{3i\omega}{2}\gamma^0\,, \end{aligned} \tag{57}$$

and the relativistic scalar product is defined as

$$\langle\psi,\psi'\rangle = \int_D d^3x\, e^{3\omega t}\overline{\psi}(x)\gamma^0\psi'(x)\,. \tag{58}$$

The mode expansion in momentum basis

$$\begin{aligned} \psi(t,\vec{x}) &= \psi^{(+)}(t,\vec{x}) + \psi^{(-)}(t,\vec{x}) \\ &= \int d^3p\sum_\sigma\left[U_{\vec{p},\sigma}(x)a(\vec{p},\sigma) + V_{\vec{p},\sigma}(x)b^\dagger(\vec{p},\sigma)\right]\,. \end{aligned} \tag{59}$$

is written in terms of the fundamental spinors of positive and negative frequencies with momentum $\vec{p}$ and helicity σ that read

$$U_{\vec{p},\sigma}(t,\vec{x}) \;=\; iN \left(\begin{array}{c} \frac{1}{2}\, e^{\pi k/2} H^{(1)}_{\nu_-}(qe^{-\omega t})\, \xi_\sigma(\vec{p}) \\ \sigma\, e^{-\pi k/2} H^{(1)}_{\nu_+}(qe^{-\omega t})\, \xi_\sigma(\vec{p}) \end{array} \right) e^{i\vec{p}\cdot\vec{x}-2\omega t} \tag{60}$$

$$V_{\vec{p},\sigma}(t,\vec{x}) \;=\; iN \left(\begin{array}{c} -\sigma\, e^{-\pi k/2} H^{(2)}_{\nu_-}(qe^{-\omega t})\, \eta_\sigma(\vec{p}) \\ \frac{1}{2}\, e^{\pi k/2} H^{(2)}_{\nu_+}(qe^{-\omega t})\, \eta_\sigma(\vec{p}) \end{array} \right) e^{-i\vec{p}\cdot\vec{x}-2\omega t}\,, \tag{61}$$

where $q = p/\omega$ and $N = \frac{1}{(2\pi)^{3/2}}\sqrt{\pi q}$.

The Pauli spinors $\xi_\sigma(\vec{p})$ and $\eta_\sigma(\vec{p}) = i\sigma_2[\xi_\sigma(\vec{p})]^*$ of helicity $\sigma = \pm 1/2$ satisfy

$$\vec{\sigma}\cdot\vec{p}\;\xi_\sigma(\vec{p}) = 2p\sigma\,\xi_\sigma(\vec{p})\,, \quad \vec{\sigma}\cdot\vec{p}\;\eta_\sigma(\vec{p}) = -2p\sigma\,\eta_\sigma(\vec{p})\,. \tag{62}$$

Thus we obtain fundamental spinors which have the properties

$$P^i\,U_{\vec{p},\sigma} = p^i\,U_{\vec{p},\sigma}\,, \qquad P^i\,V_{\vec{p},\sigma} = -p^i\,V_{\vec{p},\sigma}\,, \tag{63}$$

$$W\,U_{\vec{p},\sigma} = p\sigma U_{\vec{p},\sigma}\,, \qquad W\,V_{\vec{p},\sigma} = -p\sigma V_{\vec{p},\sigma}\,. \tag{64}$$

Moreover, these spinors are ***charge-conjugated*** to each other,

$$V_{\vec{p},\sigma} = (U_{\vec{p},\sigma})^c = \mathcal{C}(\overline{U}_{\vec{p},\sigma})^T\,, \quad \mathcal{C} = i\gamma^2\gamma^0\,, \tag{65}$$

satisfy the ortonormalization relations

$$\langle U_{\vec{p},\sigma}, U_{\vec{p}',\sigma'}\rangle = \langle V_{\vec{p},\sigma}, V_{\vec{p}',\sigma'}\rangle = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\,, \tag{66}$$

$$\langle U_{\vec{p},\sigma}, V_{\vec{p}',\sigma'}\rangle = \langle V_{\vec{p},\sigma}, U_{\vec{p}',\sigma'}\rangle = 0\,, \tag{67}$$

and represent a ***complete*** system of solutions,

$$\int d^3p \sum_{\sigma}\left[U_{\vec{p},\sigma}(t,\vec{x})U^{+}_{\vec{p},\sigma}(t,\vec{x}\,') + V_{\vec{p},\sigma}(t,\vec{x})V^{+}_{\vec{p},\sigma}(t,\vec{x}\,')\right] = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,. \tag{68}$$

The massless Dirac field

In the case of $m = 0$ the fundamental solutions of the left-handed massless Dirac field read

$$\begin{aligned} U^0_{\vec{p},\sigma}(t_c,\vec{x}) &= \lim_{k\to 0}\frac{1-\gamma^5}{2}U_{\vec{p},\sigma}(t_c,\vec{x}) \\ &= \left(\frac{-\omega t_c}{2\pi}\right)^{3/2}\begin{pmatrix} (\frac{1}{2}-\sigma)\xi_\sigma(\vec{p}) \\ 0 \end{pmatrix} e^{-ipt_c+i\vec{p}\cdot\vec{x}} \qquad (69) \\ V^0_{\vec{p},\sigma}(t_c,\vec{x}) &= \lim_{k\to 0}\frac{1-\gamma^5}{2}V_{\vec{p},\sigma}(t_c,\vec{x}) \\ &= \left(\frac{-\omega t_c}{2\pi}\right)^{3/2}\begin{pmatrix} (\frac{1}{2}+\sigma)\eta_\sigma(\vec{p}) \\ 0 \end{pmatrix} e^{ipt_c-i\vec{p}\cdot\vec{x}}\,, \qquad (70) \end{aligned}$$

are non-vanishing only for positive frequency and $\sigma = -1/2$ or negative frequency and $\sigma = 1/2$, as in Minkowski spacetime. Obviously, these solutions have similar properties as (65)-(64).

Canonical quantization

The particle (a, $a^\dagger$) and antiparticle (b, $b^\dagger$) operators fulfill the standard anticommutation relations in the momentum representation,

$$\{a(\vec{p},\sigma), a^\dagger(\vec{p}\,', \sigma')\} = \{b(\vec{p},\sigma), b^\dagger(\vec{p}\,', \sigma')\} = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}\,')\,, \tag{71}$$

since then the equal-time anticommutator takes the ***canonical*** form

$$\{\psi(t,\vec{x}),\ \overline{\psi}(t,\vec{x}\,')\} = e^{-3\omega t}\gamma^0\delta^3(\vec{x}-\vec{x}\,')\,. \tag{72}$$

In general, the propagators are constructed using the partial anticommutator functions,

$$\tilde{S}^{(\pm)}(t,t',\vec{x}-\vec{x}\,') = i\{\psi^{(\pm)}(t,\vec{x}),\ \overline{\psi}^{(\pm)}(t',\vec{x}\,')\}\,, \tag{73}$$

or the total one $\tilde{S} = \tilde{S}^{(+)} + \tilde{S}^{(-)}$. For example, the Feynman propagator,

$$\tilde{S}_F(t,t',\vec{x}-\vec{x}\,') = i\,\langle 0|\,T[\psi(x)\overline{\psi}(x')]\,|0\rangle \tag{74}$$

$$= \theta(t-t')\tilde{S}^{(+)}(t,t',\vec{x}-\vec{x}\,') - \theta(t'-t)\tilde{S}^{(-)}(t,t',\vec{x}-\vec{x}\,')\,, \tag{75}$$

is the ***causal*** Green function which obeys

$$[E_D(x)-m]\tilde{S}_F(t,t',\vec{x}-\vec{x}\,') = -e^{-3\omega t}\delta^4(x-x')\,. \tag{76}$$

The set of commuting operators of the momentum basis

The one-particle operators which are diagonal in the momentum basis are the momentum

$$\mathbf{P}^i =: \left\langle \psi, P^i \psi \right\rangle := \int d^3p\, p^i \sum_\sigma \left[a^\dagger(\vec{p},\sigma) a(\vec{p},\sigma) + b^\dagger(\vec{p},\sigma) b(\vec{p},\sigma) \right] \,, \qquad (77)$$

the helicity (or Pauli-Lubanski) operator,

$$\mathbf{W} =: \langle \psi, W\psi \rangle := \int d^3p \sum_\sigma p\sigma \left[a^\dagger(\vec{p},\sigma) a(\vec{p},\sigma) + b^\dagger(\vec{p},\sigma) b(\vec{p},\sigma) \right] \,, \qquad (78)$$

and the charge operator

$$\mathbf{Q} =: \langle \psi, \psi \rangle := \int d^3p \sum_\sigma \left[a^\dagger(\vec{p},\sigma) a(\vec{p},\sigma) - b^\dagger(\vec{p},\sigma) b(\vec{p},\sigma) \right] \,. \qquad (79)$$

Thus the momentum basis of the Fock space is formed by the common eigenvectors of the set $\{\mathbf{Q}, \mathbf{P}^i, \mathbf{W}\}$.

The Hamiltonian operator

The Hamiltonian operator $\mathbf{H} =: \langle\psi, H\psi\rangle :$ is conserved but is not diagonal in this basis since it does not commute with $\mathbf{P}^i$ and $\mathbf{W}$. Its form in momentum representation can be calculated using the identity

$$H\, U_{\vec{p},\sigma}(t,\vec{x}) = -i\omega\left(p^i\partial_{p^i} + \frac{3}{2}\right) U_{\vec{p},\sigma}(t,\vec{x})\,, \tag{80}$$

and the similar one for $V_{\vec{p},\sigma}$, leading to

$$\mathbf{H} = \frac{i\omega}{2}\int d^3p\, p^i \sum_{\sigma}\left[a^\dagger(\vec{p},\sigma)\overleftrightarrow{\partial}_{p^i} a(\vec{p},\sigma) + b^\dagger(\vec{p},\sigma)\overleftrightarrow{\partial}_{p^i} b(\vec{p},\sigma)\right] \tag{81}$$

where the derivatives act as $f\overleftrightarrow{\partial}h = f\partial h - (\partial f)h$.

These new properties are ***universal***, since similar formulas hold for the scalar and vector fields.

Concluding remarks

1. The energy operator is well-defined wherever measurements can be done.

2. The quantum modes are globally defined by complete systems of commuting operators on the whole observer's domain.

3. The charge conjugation is point-independent and, therefore, the vacuum is stable.

4. Consequently, the modes created by the same c.s.c.o. in different charts can be related among themselves through combined transformations without to mix the particle and antiparticle subspaces.

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