PERTURBATIVE Q.E.D
ON DE SITTER UNIVERSE

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THE THEORY OF FREE FIELDS ON DE SITTER UNIVERSE

Dirac field

• De Sitter metric:

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x}^{\,2} = \frac{1}{(\omega t_c)^2}(dt_c^2 - d\vec{x}^{\,2}), \quad \omega > 0, \quad \omega t_c = -e^{-\omega t} \tag{1}$$

$$g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_{\mu} e^{\hat{\beta}}_{\nu} \tag{2}$$

• In the Cartesian gauge the nonvanishing tetradic components are:

$$e^{0}_{\hat{0}} = -\omega t_c, \quad e^{i}_{\hat{j}} = -\delta^{i}_{\hat{j}} \omega t_c, \quad e^{\hat{0}}_{0} = -\frac{1}{\omega t_c}, \quad e^{\hat{i}}_{j} = -\delta^{\hat{i}}_{j} \frac{1}{\omega t_c}. \tag{3}$$

• Action for Dirac field

$$\mathcal{S}\,[e,\psi] \;=\; \int d^4x\sqrt{-g}\Big\{\frac{i}{2}[\bar{\psi}\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi-(\overline{D_{\hat{\alpha}}\psi})\gamma^{\hat{\alpha}}\psi] \\ -m\bar{\psi}\psi\Big\}, \tag{4}$$

where

$$D_{\hat{\alpha}} = e^{\mu}_{\hat{\alpha}}D_{\mu} = \partial_{\hat{\alpha}} + \Gamma_{\hat{\alpha}} \tag{5}$$

• In this gauge the Dirac operator reads:

$$\begin{aligned} E_D \;&=\; i\gamma^0\partial_t + ie^{-\omega t}\gamma^i\partial_i + \frac{3i\omega}{2}\gamma^0 \\ &=\; -i\omega t_c(\gamma^0\partial_{t_c}+\gamma^i\partial_i) + \frac{3i\omega}{2}\gamma^0 \end{aligned} \tag{6}$$

• the Dirac equation

$$(E_D - m)\psi = 0 \tag{7}$$

• The fundamental solutions are:

$$\begin{aligned} U_{\vec{p},\,\lambda}(t,\vec{x}\,) \;&=\; i\frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}}\begin{pmatrix} \frac{1}{2}e^{\pi k/2}H^{(1)}_{\nu_-}(qe^{-\omega t})\xi_\lambda(\vec{p}\,) \\ \lambda e^{-\pi k/2}H^{(1)}_{\nu_+}(qe^{-\omega t})\xi_\lambda(\vec{p}\,) \end{pmatrix} e^{i\vec{p}\cdot\vec{x}-2\omega t} \\ V_{\vec{p},\,\lambda}(t,\vec{x}\,) \;&=\; i\frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}}\begin{pmatrix} -\lambda e^{-\pi k/2}H^{(2)}_{\nu_-}(qe^{-\omega t})\eta_\lambda(\vec{p}\,) \\ \frac{1}{2}e^{\pi k/2}H^{(2)}_{\nu_+}(qe^{-\omega t})\eta_\lambda(\vec{p}\,) \end{pmatrix} e^{-i\vec{p}\cdot\vec{x}-2\omega t}\,, \end{aligned} \tag{8}$$

• Charge conjugation

$$U_{\vec{p},\,\lambda}(x) \to V_{\vec{p},\,\lambda}(x) = i\gamma^2\gamma^0(\bar{U}_{\vec{p},\,\lambda}(x))^T. \tag{9}$$

• The orthonormalization relations :

$$\begin{aligned}\int d^3x(-g)^{1/2}\bar{U}_{\vec{p},\,\lambda}(x)\gamma^0 U_{\vec{p}\,',\,\lambda'}(x) &= \\ \int d^3x(-g)^{1/2}\bar{V}_{\vec{p},\,\lambda}(x)\gamma^0 V_{\vec{p}\,',\,\lambda'}(x) &= \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,') \\ \int d^3x(-g)^{1/2}\bar{U}_{\vec{p},\lambda}(x)\gamma^0 V_{\vec{p}\,',\lambda'}(x) &= 0,\end{aligned} \tag{10}$$

$$\int d^3p \sum_{\lambda}\left[U_{\vec{p},\,\lambda}(t,\vec{x}\,)U^{+}_{\vec{p},\,\lambda}(t,\vec{x}\,') + V_{\vec{p},\,\lambda}(t,\vec{x}\,)V^{+}_{\vec{p},\,\lambda}(t,\vec{x}\,')\right] = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,'). \tag{11}$$

• The field operator

$$\begin{aligned}\psi(t,\vec{x}\,) &= \psi^{(+)}(t,\vec{x}\,) + \psi^{(-)}(t,\vec{x}\,) \\ &= \int d^3p \sum_{\lambda}\left[U_{\vec{p},\,\lambda}(t,\vec{x}\,)a(\vec{p},\,\lambda) + V_{\vec{p},\,\lambda}(t,\vec{x}\,)b^{+}(\vec{p},\,\lambda)\right].\end{aligned} \tag{12}$$

• Canonic cuantization

$$\{a(\vec{p},\,\lambda), a^{+}(\vec{p}\,',\lambda')\} = \{b(\vec{p},\,\lambda), b^{+}(\vec{p}\,',\lambda')\} = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,')\,, \tag{13}$$

$$\{\psi(t,\vec{x}\,),\overline{\psi}(t,\vec{x}\,'\,)\}=e^{-3\omega t}\gamma^0\delta^3(\vec{x}-\vec{x}\,'\,)\;, \tag{14}$$

$$\begin{aligned} S_F(t,t',\vec{x}-\vec{x}\,'\,) &= i\langle 0|T[\psi(x)\overline{\psi}(x')]|0\rangle \\ &= \theta(t-t')S^{(+)}(t,t',\vec{x}-\vec{x}\,'\,) \\ &\quad -\theta(t'-t)S^{(-)}(t,t',\vec{x}-\vec{x}\,'\,)\;. \end{aligned} \tag{15}$$

$$[E_D(x)-m]S_F(t,t',\vec{x}-\vec{x}\,'\,)=-e^{-3\omega t}\delta^4(x-x'). \tag{16}$$

- Conserved operators

$$\mathbf{P}^i =: \langle\psi, P^i\psi\rangle := \int d^3p\, p^i \sum_\lambda [a^+(\vec{p},\,\lambda)a(\vec{p},\,\lambda)+b^+(\vec{p},\,\lambda)b(\vec{p},\,\lambda)] \tag{17}$$

$$\mathbf{W} =: \langle\psi, W\psi\rangle := \int d^3p\, \sum_\lambda p\lambda[a^+(\vec{p},\,\lambda)a(\vec{p},\,\lambda)+b^+(\vec{p},\,\lambda)b(\vec{p},\,\lambda)]\;. \tag{18}$$

$$\mathbf{H}=\frac{i\omega}{2}\int d^3p\, p^i \sum_\lambda \left[a^+(\vec{p},\lambda)\stackrel{\leftrightarrow}{\partial}_{p^i} a(\vec{p},\lambda)+b^+(\vec{p},\lambda)\stackrel{\leftrightarrow}{\partial}_{p^i} b(\vec{p},\lambda)\right]. \tag{19}$$

•

$$[H, P^i] = i\omega P^i. \tag{20}$$

Scalar field

• Action of the scalar field

$$\mathcal{S}[\phi, \phi^*] = \int d^4x \sqrt{-g}\, \mathcal{L} = \int d^4x \sqrt{-g}\, \left(\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi\right) \ , \tag{21}$$

• The Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\, \partial_\mu \left[\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi\right] + m^2 \phi = 0\,. \tag{22}$$

$$\left(\partial_t^2 - e^{-2\omega t} \Delta + 3\omega \partial_t + m^2\right) \phi(x) = 0\,. \tag{23}$$

• Solutions of Klein-Gordon equation:

$$f_{\vec{p}}(x) = \frac{1}{2} \sqrt{\frac{\pi}{\omega}}\, \frac{e^{-3\omega t/2}}{(2\pi)^{3/2}}\, e^{-\pi k/2} H_{ik}^{(1)} \left(\frac{p}{\omega}\, e^{-\omega t}\right) e^{i\vec{p}\cdot\vec{x}} \ , \tag{24}$$

$$k = \sqrt{\mu^2 - \tfrac{9}{4}}\,, \quad \mu = \frac{m}{\omega}\,, m > 3\omega/2 \tag{25}$$

- The orthonormalization relations:

$$\begin{aligned} i\int d^3x\,(-g)^{1/2}\, f^*_{\vec{p}}(x)\, \overset{\leftrightarrow}{\partial_t}\, f_{\vec{p}\,'}(x) \;&=\; \\ -i\int d^3x\,(-g)^{1/2}\, f_{\vec{p}}(x)\, \overset{\leftrightarrow}{\partial_t}\, f^*_{\vec{p}\,'}(x) \;&=\; \delta^3(\vec{p}-\vec{p}\,')\,, \end{aligned} \tag{26}$$

$$i\int d^3x\,(-g)^{1/2}\, f_{\vec{p}}(x)\, \overset{\leftrightarrow}{\partial_t}\, f_{\vec{p}\,'}(x) \;=\; 0\,, \tag{27}$$

$$i\int d^3p\, f^*_{\vec{p}}(t,\vec{x}\,)\, \overset{\leftrightarrow}{\partial_t}\, f_{\vec{p}}(t,\vec{x}\,') = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,. \tag{28}$$

- Canonic cuantization

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) = \int d^3p\left[f_{\vec{p}}(x)a(\vec{p}) + f^*_{\vec{p}}(x)b^+(\vec{p})\right], \tag{29}$$

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = [b(\vec{p}), b^\dagger(\vec{p}\,')] = \delta^3(\vec{p}-\vec{p}\,')\,. \tag{30}$$

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$$[\phi(t,\vec{x}\,), \partial_t\phi^\dagger(t,\vec{x}\,')] = ie^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,. \tag{31}$$

• The commutator functions:

$$D^{(\pm)}(x,x') = i[\phi^{(\pm)}(x), \phi^{(\pm)\dagger}(x')], \tag{32}$$

$$\begin{aligned} D^{(+)}(x,x') &= i\int d^3p\, f_{\vec{p}}\,(x) f^*_{\vec{p}}(x'), \\ D^{(-)}(x,x') &= -i\int d^3p\, f^*_{\vec{p}}(x) f_{\vec{p}}\,(x'). \end{aligned} \tag{33}$$

• $G(x,x')$ is a Green function of the Klein-Gordon equation if:

$$\left(\partial_t^2 - e^{-2\omega t}\Delta_x + 3\omega\partial_t + m^2\right) G(x,x') = e^{-3\omega t}\delta^4(x-x')\,. \tag{34}$$

$$D_R(t,t',\vec{x}-\vec{x}\,') = \theta(t-t')D(t,t',\vec{x}-\vec{x}\,')\,, \tag{35}$$

$$D_A(t,t',\vec{x}-\vec{x}\,') = -\,\theta(t'-t)D(t,t',\vec{x}-\vec{x}\,')\,, \tag{36}$$

• The Feynman propagator,

$$\begin{aligned} D_F(t,t',\vec{x}-\vec{x}\,') &= i\langle 0|T[\phi(x)\phi^\dagger(x')]\,|0\rangle \\ &= \theta(t-t')D^{(+)}(t,t',\vec{x}-\vec{x}\,') - \theta(t'-t)D^{(-)}(t,t',\vec{x}-\vec{x}\,')\,, \end{aligned} \tag{37}$$

The electromagnetic field

• The action

$$\mathcal{S}[A] = \int d^4x\sqrt{-g}\,\mathcal{L} = -\frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu}, \tag{38}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

• Field equation

$$\partial_\nu(\sqrt{-g}\,g^{\nu\alpha}g^{\mu\beta}F_{\alpha\beta}) = 0\,, \tag{39}$$

• In the Coulomb gauge $A_0 = 0$, $(A^i)_{\,;i} = 0$,and chart $\{t_c, \vec{x}\}$

$$(\partial_{t_c}^2 - \Delta)A_i = 0 \tag{40}$$

• The field operator

$$A_i(x) = A_i^{(+)}(x) + A_i^{(-)}(x)$$
$$= \int d^3k \sum_\lambda \left[e_i(\vec{n}_k,\lambda) f_{\vec{k}}(x) a(\vec{k},\lambda) + [e_i(\vec{n}_k,\lambda) f_{\vec{k}}(x)]^* a^*(\vec{k},\lambda) \right] \tag{41}$$

$$f_{\vec{k}}(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}}\, e^{-ikt_c + i\vec{k}\cdot\vec{x}}\,, \tag{42}$$

• the Hermitian form

$$\langle f, g\rangle = i\int d^3x\, f^*(t_c,\vec{x}\,)\, \overset{\leftrightarrow}{\partial}_{t_c}\, g(t_c,\vec{x}\,)\,, \quad f\, \overset{\leftrightarrow}{\partial}\, g = f(\partial g) - g(\partial f) \tag{43}$$

• The orthonormalization relations:

$$\langle f_{\vec{k}}\,, f_{\vec{k}\,'}\rangle = -\langle f^*_{\vec{k}}\,, f^*_{\vec{k}'}\rangle = \delta^3(\vec{k} - \vec{k}\,')\,, \tag{44}$$
$$\langle f_{\vec{k}}\,, f^*_{\vec{k}\,'}\rangle = 0\,, \tag{45}$$

$$i\int d^3k\, f^*_{\vec{k}}(t_c,\vec{x})\, \overset{\leftrightarrow}{\partial}_{t_c}\, f_{\vec{k}}(t_c,\vec{x}\,') = \delta^3(\vec{x} - \vec{x}\,')\,, \tag{46}$$

$$\vec{k}\cdot\vec{e}(\vec{n}_k,\lambda) = 0\,, \tag{47}$$

$$\vec{e}\,(\vec{n}_k,\lambda)\cdot\vec{e}\,(\vec{n}_k,\lambda')^* \;=\; \delta_{\lambda\lambda'}\,, \tag{48}$$

$$\sum_{\lambda} e_i(\vec{n}_k,\lambda)\,e_j(\vec{n}_k,\lambda)^* \;=\; \delta_{ij}-\frac{k^ik^j}{k^2}\,. \tag{49}$$

- In local frame

$$A_{\hat{i}}\,(x)=e\,_{\hat{i}}{}^{\cdot j}_{\cdot}A_j=\int d^3k\sum_{\lambda}\left[w_{\hat{i}(\vec{k},\lambda)}(x)a(\vec{k},\lambda)+[w_{\hat{i}(\vec{k},\lambda)}(x)]^*a^*(\vec{k},\lambda)\right]. \tag{50}$$

- Conformal transformation

$$g_{\mu\nu}(x)=\Omega(x)\eta_{\mu\nu}, \tag{51}$$

$$g^{\mu\nu}(x)=\Omega^{-1}(x)\eta^{\mu\nu},\quad ds^2=\Omega\,ds_c^2, \tag{52}$$

$$A_\mu=A'_\mu\;\;;\quad A^\mu=\Omega^{-1}A'^\mu. \tag{53}$$

- Quantization

$$[a(\vec{k},\lambda),a^\dagger(\vec{k}\,',\lambda')]=\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}\,'\,)\,. \tag{54}$$

$$[A_i(t_c,\vec{x}\,),\pi^j(t_c,\vec{x}\,'\,)]=[A_i(t_c,\vec{x}\,),\partial_{t_c}A_j(t_c,\vec{x}\,'\,)]=i\,\delta^{tr}_{ij}(\vec{x}-\vec{x}\,'\,)\,, \tag{55}$$

• The momentum density

$$\pi^j = \sqrt{-g}\,\frac{\delta\mathcal{L}}{\delta(\partial_{t_c}A_j)} = \partial_{t_c}A_j \tag{56}$$

$$\delta^{tr}_{ij}(\vec{x}\,) = \frac{1}{(2\pi)^3}\int d^3q\left(\delta_{ij} - \frac{q^i q^j}{q^2}\right)e^{i\vec{q}\cdot\vec{x}} \tag{57}$$

• Conserved operators

$$\mathcal{P}^l = \frac{1}{2}\,\delta_{ij} : \langle A_i, (P^l A)_j\rangle := \int d^3k\, k^l \sum_\lambda a^\dagger(\vec{k},\lambda)a(\vec{k},\lambda)\,, \tag{58}$$

$$\mathcal{H} = \frac{1}{2}\,\delta_{ij} : \langle A_i, (HA)_j\rangle : \tag{59}$$

$$\mathcal{W} \;=\; \frac{1}{2}\,\delta_{ij} : \langle A_i, (WA)_j\rangle := \int d^3k\, k \sum_\lambda \,\lambda\, a^\dagger(\vec{k},\lambda)a(\vec{k},\lambda) \tag{60}$$

$$[\mathcal{H},\mathcal{P}^i]=i\omega\mathcal{P}^i\,,\quad[\mathcal{H},\mathcal{W}]=i\omega\mathcal{W}\,,\quad[\mathcal{W},\mathcal{P}^i]=0\,, \tag{61}$$

$$\left(H f_{\vec{k}}\right)(x)=-i\omega\left(k^i\partial_{k_i}+\frac{3}{2}\right)f_{\vec{k}}\,(x) \tag{62}$$

$$\mathcal{H}=\frac{i\omega}{2}\int d^3k\,k^i\left[a^\dagger(\vec{k},\lambda)\overleftrightarrow{\partial}_{k_i}\,a(\vec{k},\lambda)\right]\,, \tag{63}$$

- The commutator functions

$$D^{(\pm)}_{ij}(x-x')=i[A^{(\pm)}_i(x),A^{(\pm)\,\dagger}_j(x')] \tag{64}$$

$$D^{(+)}_{ij}(x-x')\;=\;i\int d^3k\,f_{\vec{k}}\,(x)f_{\vec{k}}\,(x\,')^*\left(\delta_{ij}-\frac{k^ik^j}{k^2}\right) \tag{65}$$

$$D^{(+)}_{ij}(x-x')=\frac{i}{(2\pi)^3}\int\frac{d^3k}{2k}\left(\delta_{ij}-\frac{k^ik^j}{k^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')-ik(t_c-t'_c)} \tag{66}$$

• at equal times,

$$\partial_{t_c} D_{ij}^{(+)}(t_c - t'_c,\, \vec{x} - \vec{x}\,')\Big|_{t'_c = t_c} = \frac{1}{2}\,\delta_{ij}^{tr}(\vec{x} - \vec{x}\,')\,. \tag{67}$$

• The transversal Green functions

$$\left(\partial_{t_c}^2 - \Delta_x\right) G_{ij}(x - x') = \delta(t_c - t'_c)\delta_{ij}^{tr}(\vec{x} - \vec{x}\,') \tag{68}$$

şi $\partial_i G^{i\cdot}_{\cdot j}(x) = 0$.

$$D_{ij}^{R}(x - x') \;=\; \theta(t_c - t'_c) D_{ij}(x - x')\,, \tag{69}$$

$$D_{ij}^{A}(x - x') \;=\; -\theta(t'_c - t_c) D_{ij}(x - x')\,, \tag{70}$$

• The Feynman propagator ,

$$\begin{aligned} &D_{ij}^{F}(x - x') = i\langle 0|T[A_i(x)A_j(x')]\,|0\rangle \\ &= \theta(t_c - t'_c) D_{ij}^{(+)}(x - x') - \theta(t'_c - t_c) D_{ij}^{(-)}(x - x')\,, \end{aligned} \tag{71}$$

Interaction between Dirac field and electromagnetic field

• The action for interacting fields:

$$\mathcal{S}\,[e,\psi,A] \;=\; \int d^4x\sqrt{-g}\Big\{\frac{i}{2}[\bar{\psi}\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi-(\overline{D_{\hat{\alpha}}\psi})\gamma^{\hat{\alpha}}\psi] \\ -m\bar{\psi}\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-e\bar{\psi}\gamma^{\hat{\alpha}}A_{\hat{\alpha}}\psi\Big\}, \tag{72}$$

• Equations for fields in interaction

$$\begin{aligned} i\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi-m\psi \;&=\; e\gamma^{\hat{\alpha}}A_{\hat{\alpha}}\psi, \\ \frac{1}{\sqrt{(-g)}}\,\partial_\mu\left(\sqrt{(-g)}F^{\mu\nu}\right) \;&=\; ee^{\nu}_{\hat{\alpha}}\bar{\psi}\gamma^{\hat{\alpha}}\psi. \end{aligned} \tag{73}$$

Interaction between scalar field and electromagnetic field

● The action

$$\mathcal{S}[\phi, A] = \int d^4x\sqrt{-g}\Big\{(\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$-ie[(\partial_\mu\phi^\dagger)\phi - (\partial_\mu\phi)\phi^\dagger]A^\mu + e^2\phi^\dagger\phi A_\mu A^\mu\Big\}. \quad (74)$$

● Equations for fields in interaction

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi - ie\sqrt{-g}\phi A^\mu\right] + m^2\phi = ie(\partial_\mu\phi)A^\mu + e^2\phi A_\mu A^\mu,$$
$$\frac{1}{\sqrt{(-g)}}\partial_\mu\left(\sqrt{(-g)}F^{\mu\nu}\right) = -ie[(\partial^\nu\phi^\dagger)\phi - (\partial^\nu\phi)\phi^\dagger]$$
$$+2e^2\phi^\dagger\phi A^\nu. \quad (75)$$

The in out fields

- The equation of the interaction fields in the Coulomb gauge

$$(\partial_{t_c}^2 - \Delta_x)\vec{A}(x) = e\bar{\psi}(x)\vec{\gamma}\psi(x) \tag{76}$$

- Solution:

$$\vec{A}(x) = \vec{\hat{A}}(x) - e\int d^3y dt'_c\, D^G(x-y)\bar{\psi}(y)\vec{\gamma}\psi(y), \tag{77}$$

$$\vec{A}(x) = \vec{\hat{A}}_{(R/A)}(x) - e\int d^3y dt'_c\, D_{R/A}(x-y)\bar{\psi}(y)\vec{\gamma}\psi(y), \tag{78}$$

- the free fields , $\vec{\hat{A}}_{(R/A)}(x)$ satisfy:

$$\lim_{t\to\mp\infty}(\vec{A}(x) - \vec{\hat{A}}_{(R/A)}(x)) = 0. \tag{79}$$

- The fields $\mathbf{in}/\mathbf{out}$:

$$\sqrt{z_3}\vec{A}_{in/out}(x) \;=\; \vec{A}(x) + e\int d^3y dt'_c\, D_{R/A}(x-y)\bar{\psi}(y)\vec{\gamma}\psi(y). \tag{80}$$

• The $\mathbf{in/out}$ fields, final expressions:

$$\sqrt{z_3}\vec{A}_{in/out}(x) = \vec{A}(x) + \int d^3y dt'_c\, D_{R/A}(x-y)(\partial^2_{t'_c} - \Delta_y)\vec{A}(y), \tag{81}$$

$$a(\vec{k},\lambda)_{in/out} = i\int d^3x f_{\vec{k}}(x)\vec{e}(\vec{n}_k,\lambda)\overleftrightarrow{\partial}_{t_c}\vec{A}_{in/out}(x) \tag{82}$$

• The case of interaction with scalar field

$$\begin{aligned}\phi(x) &= \hat{\phi}(x) + e\int d^4y\sqrt{-g}\,G(x,y)\Big\{\frac{i}{\sqrt{-g}}\partial_\mu[\sqrt{-g}\phi(y)A^\mu(y)] \\ &\quad + i[\partial_\mu\phi(y)]A^\mu(y) + e\phi(y)A_\mu(y)A^\mu(y)\Big\},\end{aligned} \tag{83}$$

$$\begin{aligned}\phi(x) &= \hat{\phi}_{R/A}(x) + e\int d^4y\sqrt{-g}D_{R/A}(x,y)\Big\{\frac{i}{\sqrt{-g}}\partial_\mu[\sqrt{-g}\phi(y)A^\mu(y)] \\ &\quad + i[\partial_\mu\phi(y)]A^\mu(y) + e\phi(y)A_\mu(y)A^\mu(y)\Big\}.\end{aligned} \tag{84}$$

• The $\mathbf{in}/\mathbf{out}$ fields:

$$\sqrt{z}\phi_{in/out}(x) = \phi(x) - \int d^4y\sqrt{-g}D_{R/A}(x,y)[E_{KG}(y) + m^2]\phi(y). \tag{85}$$

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$$\begin{aligned} a(\vec{p})_{in/out} &= i\int d^3x(-g)^{1/2}f^*_{\vec{p}}(x) \overleftrightarrow{\partial_t} \phi_{in/out}(x) \\ b^\dagger(\vec{p})_{in/out} &= i\int d^3x(-g)^{1/2}f_{\vec{p}}(x) \overleftrightarrow{\partial_t} \phi_{in/out}(x). \end{aligned} \tag{86}$$

Perturbation theory

• The expression of Green functions

$$G(y_1, y_2, ..., y_n) = \frac{1}{\langle 0|\widetilde{S}|0\rangle}\langle 0|T[\hat{\phi}(y_1)\hat{\phi}(y_2)...\hat{\phi}(y_n), \widetilde{S}]|0\rangle, \tag{87}$$

$$\widetilde{S} = Te^{-i\int \mathcal{L}_I(x)d^4x} = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!}\int T[\mathcal{L}_I(x_1)...\mathcal{L}_I(x_n)]d^4x_1...d^4x_n, \tag{88}$$

• The interaction Lagrangian

$$\mathcal{L}_I(x) = -ie\sqrt{-g} : [(\partial_\mu\phi^\dagger)\phi - (\partial_\mu\phi)\phi^\dagger]A^\mu : \tag{89}$$

$$\frac{(-i)^n}{n!\langle 0|\widetilde{S}|0\rangle}\int \langle 0|T[\hat{\phi}(y_1)\hat{\phi}(y_2)...\hat{\phi}(y_n), \mathcal{L}_I(x_1)...\mathcal{L}_I(x_n)]|0\rangle d^4x_1...d^4x_n. \tag{90}$$

• the amplitude:

$$\langle out, 1(\vec{p}\,')|in, 1(\vec{p})\rangle = \delta^3(\vec{p}-\vec{p}\,') - \frac{1}{z}\int\int\sqrt{-g(y)}\sqrt{-g(z)}f^*_{\vec{p}\,'}(y)[E_{KG}(y)+m^2] \times\langle 0|T[\phi(y)\phi^\dagger(z)]|0\rangle[E_{KG}(z)+m^2]f_{\vec{p}}(z)d^4yd^4z. \quad (91)$$

• The scattering amplitude in the first order of perturbation theory

$$A_{i\to f} = -e\int\sqrt{-g(x)}\left[f^*_{\vec{p}\,'}(x)\overleftrightarrow{\partial_\mu} f_{\vec{p}}(x)\right]A^\mu(x)d^4x, \quad (92)$$

$$A_{i\to f} = -ie\int\sqrt{-g(x)}\bar{U}_{\vec{p}\,',\lambda'}(x)\gamma_\mu A^{\hat{\mu}}(x)U_{\vec{p},\lambda}(x)d^4x, \quad (93)$$

Coulomb scattering for the charged scalar field

• External field

$$A^{\hat{0}} = \frac{Ze}{|\vec{x}|}e^{-\omega t} \tag{94}$$

• initial and final states are:

$$\phi_f(x) = f_{\vec{p}_f}(x), \quad \phi_i(x) = f_{\vec{p}_i}(x). \tag{95}$$

• Scattering amplitude

$$\begin{aligned} A_{i\to f} \; &= \; \frac{-\alpha Z}{8\pi|\vec{p_f} - \vec{p_i}|^2}\Bigg\{-\frac{p_i}{2}\int_0^\infty z\left[H_{ik}^{(2)}(p_f z)H_{ik-1}^{(1)}(p_i z) - H_{ik}^{(2)}(p_f z)H_{ik+1}^{(1)}(p_i z)\right]dz \\ &+\frac{p_f}{2}\int_0^\infty z\left[H_{ik-1}^{(2)}(p_f z)H_{ik}^{(1)}(p_i z) - H_{ik+1}^{(2)}(p_f z)H_{ik}^{(1)}(p_i z)\right]dz\Bigg\}, \end{aligned} \tag{96}$$

$$z = \frac{e^{-\omega t}}{\omega}. \tag{97}$$

● Scattering amplitude in final form

$$A_{i\to f} = \frac{-\alpha Z}{8\pi|\vec{p_f}-\vec{p_i}|^2}B_k, \tag{98}$$

$$\begin{aligned} B_k = & \frac{(p_f+p_i)}{\sqrt{p_f p_i}}2i\delta(p_f-p_i) + \frac{1}{p_i}\theta(p_i-p_f)\left[h_k^*\left(\frac{p_f}{p_i}\right) - g_k\left(\frac{p_f}{p_i}\right)\right] \\ & + \frac{1}{p_f}\theta(p_f-p_i)\left[g_k^*\left(\frac{p_i}{p_f}\right) - h_k\left(\frac{p_i}{p_f}\right)\right]. \end{aligned} \tag{99}$$

$$\begin{aligned} g_k(\chi) = & \frac{e^{-\pi k}\chi^{ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{3}{2},\frac{1}{2}+ik;1+ik;\chi^2\right)}{B(\frac{1}{2}-ik;1+ik)} - \frac{{}_2F_1\left(\frac{1}{2},\frac{3}{2}+ik;1+ik;\chi^2\right)}{B(-\frac{1}{2}-ik;1+ik)}\right] \\ & + \frac{e^{\pi k}\chi^{-ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{3}{2},\frac{1}{2}-ik;1-ik;\chi^2\right)}{B(\frac{1}{2}+ik;1-ik)} - \frac{{}_2F_1\left(\frac{1}{2},\frac{3}{2}-ik;1-ik;\chi^2\right)}{B(-\frac{1}{2}+ik;1-ik)}\right] \\ & + \frac{\chi^{ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{3}{2},\frac{1}{2}+ik;1+ik;\chi^2\right)}{B(-\frac{1}{2};1+ik)} - \frac{{}_2F_1\left(\frac{1}{2},\frac{3}{2}+ik;1+ik;\chi^2\right)}{B(\frac{1}{2};1+ik)}\right] \\ & + \frac{\chi^{-ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{3}{2},\frac{1}{2}-ik;1-ik;\chi^2\right)}{B(-\frac{1}{2};1-ik)} - \frac{{}_2F_1\left(\frac{1}{2},\frac{3}{2}-ik;1-ik;\chi^2\right)}{B(\frac{1}{2};1-ik)}\right], \end{aligned} \tag{100}$$

$$
\begin{aligned}
h_k(\chi) \;=\; & \frac{e^{-\pi k}\chi^{-ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{3}{2},\frac{3}{2}-ik;2-ik;\chi^2\right)}{B(-\frac{1}{2}+ik;2-ik)}\cdot\chi^2-\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2}-ik;-ik;\chi^2\right)}{B(-ik;\frac{1}{2}+ik)}\right]\\
&+\frac{e^{\pi k}\chi^{ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{3}{2},\frac{3}{2}+ik;2+ik;\chi^2\right)}{B(-\frac{1}{2}-ik;2+ik)}\cdot\chi^2-\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2}+ik;ik;\chi^2\right)}{B(\frac{1}{2}-ik;ik)}\right]\\
&+\frac{\chi^{ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2}+ik;ik;\chi^2\right)}{B(\frac{1}{2};ik)}-\frac{{}_2F_1\left(\frac{3}{2},\frac{3}{2}+ik;2+ik;\chi^2\right)}{B(-\frac{1}{2};2+ik)}\cdot\chi^2\right]\\
&+\frac{\chi^{-ik}}{\sinh^2(\pi k)}\left[\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2}-ik;-ik;\chi^2\right)}{B(\frac{1}{2};-ik)}-\frac{{}_2F_1\left(\frac{3}{2},\frac{3}{2}-ik;2-ik;\chi^2\right)}{B(-\frac{1}{2};2-ik)}\cdot\chi^2\right],
\end{aligned}
\tag{101}
$$

- the notations are $\chi = \frac{p_f}{p_i}$ or $\frac{p_i}{p_f}$.

- The Minkowski limit

$$B_\infty = \frac{(p_f + p_i)}{\sqrt{p_f p_i}} 2i\delta(p_f - p_i). \tag{102}$$

- Minkowski case

$$B_M = \frac{(E_f + E_i)}{\sqrt{E_f E_i}} 4i\delta(E_f - E_i), \tag{103}$$

• Ration of the two amplitudes

$$\frac{\delta(p_f - p_i)}{2\delta(E_f - E_i)} = \frac{1}{2}. \tag{104}$$

• The incident flux

$$J_\mu = -i\left[f^*_{\vec{p}}(x)\overset{\leftrightarrow}{\partial}_\mu f_{\vec{p}}(x)\right], \tag{105}$$

$$j(t) = -i\left[f^*_{\vec{p}_i}(x)\overset{\leftrightarrow}{\partial}_i f_{\vec{p}_i}(x)\right] = \frac{\pi p_i e^{-3\omega t}}{2\omega(2\pi)^3} H^{(2)}_{ik}\left(\frac{p_i}{\omega}e^{-\omega t}\right) H^{(1)}_{ik}\left(\frac{p_i}{\omega}e^{-\omega t}\right). \tag{106}$$

Coulomb scattering with fermions

• The scattering amplitude

$$A_{i\to f} = -ie \int \sqrt{-g}\bar{U}_{\vec{p}_f,\lambda_f}(x)\gamma_0 A^{\hat{0}}(x) U_{\vec{p}_i,\lambda_i}(x) d^4x. \quad (107)$$

$$\begin{aligned} A_{i\to f} &= -i\alpha Z \frac{\sqrt{p_i p_f}}{8\pi|\vec{p}_f - \vec{p}_i|^2} \xi^+_{\lambda_f}(\vec{p}_f)\xi_{\lambda_i}(\vec{p}_i) \left[e^{\pi k} \int_0^\infty dzz H^{(2)}_{\nu_+}(p_f z) H^{(1)}_{\nu_-}(p_i z) \right. \\ &\left. + sgn(\lambda_f \lambda_i) e^{-\pi k} \int_0^\infty dzz H^{(2)}_{\nu_-}(p_f z) H^{(1)}_{\nu_+}(p_i z) \right] \end{aligned} \quad (108)$$

- Final form of the amplitude

$$
\begin{aligned}
A_{i\to f} = {} & \frac{-i}{4\pi}\frac{\alpha Z}{|\vec{p}_f-\vec{p}_i|^2}\xi^{+}_{\lambda_f}(\vec{p}_f)\xi_{\lambda_i}(\vec{p}_i)\Bigg\{\delta(p_f-p_i)+\theta(p_i-p_f)\frac{1}{p_i}\left[i\left(\frac{p_f}{p_i}\right)^{-ik}\frac{e^{\pi k}}{\cosh(\pi k)}\frac{{}_2F_1\left(\frac{1}{2},1-ik;\frac{1}{2}-ik;\frac{p_f^2}{p_i^2}\right)}{B(\frac{1}{2},\frac{1}{2}-ik)}\right.\\
& \left.-i\left(\frac{p_f}{p_i}\right)^{1+ik}\frac{e^{-\pi k}}{\cosh(\pi k)}\frac{{}_2F_1\left(\frac{3}{2},1+ik;\frac{3}{2}+ik;\frac{p_f^2}{p_i^2}\right)}{B(-\frac{1}{2},\frac{3}{2}+ik)}\right]+\theta(p_f-p_i)\frac{1}{p_f}\left[i\left(\frac{p_i}{p_f}\right)^{1-ik}\right.\\
& \left.\times\frac{e^{-\pi k}}{\cosh(\pi k)}\frac{{}_2F_1\left(\frac{3}{2},1-ik;\frac{3}{2}-ik;\frac{p_i^2}{p_f^2}\right)}{B(-\frac{1}{2},\frac{3}{2}-ik)}-\left(\frac{p_i}{p_f}\right)^{ik}\frac{ie^{\pi k}}{\cosh(\pi k)}\frac{{}_2F_1\left(\frac{1}{2},1+ik;\frac{1}{2}+ik;\frac{p_i^2}{p_f^2}\right)}{B(\frac{1}{2},\frac{1}{2}+ik)}\right]\\
& +sgn(\lambda_f\lambda_i)\delta(p_f-p_i)+sgn(\lambda_f\lambda_i)\theta(p_i-p_f)\frac{1}{p_i}\left[i\left(\frac{p_f}{p_i}\right)^{ik}\frac{e^{-\pi k}}{\cosh(\pi k)}\right.\\
& \left.\times\frac{{}_2F_1\left(\frac{1}{2},1+ik;\frac{1}{2}+ik;\frac{p_f^2}{p_i^2}\right)}{B(\frac{1}{2},\frac{1}{2}+ik)}-i\left(\frac{p_f}{p_i}\right)^{1-ik}\frac{e^{\pi k}}{\cosh(\pi k)}\frac{{}_2F_1\left(\frac{3}{2},1-ik;\frac{3}{2}-ik;\frac{p_f^2}{p_i^2}\right)}{B(-\frac{1}{2},\frac{3}{2}-ik)}\right]\\
& +sgn(\lambda_f\lambda_i)\theta(p_f-p_i)\frac{1}{p_f}\left[i\left(\frac{p_i}{p_f}\right)^{1+ik}\frac{e^{\pi k}}{\cosh(\pi k)}\frac{{}_2F_1\left(\frac{3}{2},1+ik;\frac{3}{2}+ik;\frac{p_i^2}{p_f^2}\right)}{B(-\frac{1}{2},\frac{3}{2}+ik)}\right.\\
& \left.-i\left(\frac{p_i}{p_f}\right)^{-ik}\frac{e^{-\pi k}}{\cosh(\pi k)}\frac{{}_2F_1\left(\frac{1}{2},1-ik;\frac{1}{2}-ik;\frac{p_i^2}{p_f^2}\right)}{B(\frac{1}{2},\frac{1}{2}-ik)}\right]\Bigg\}. \qquad (109)
\end{aligned}
$$

- The incident flux

$$j=\frac{e^{\omega t}}{\mid \vec{p}_i \mid}\bar{U}_{\vec{p}_i,\,\lambda_i}(x)(\vec{p}_i\vec{\gamma})U_{\vec{p}_i,\,\lambda_i}(x) \;=\; \frac{\pi p_i e^{-3\omega t}}{(2\pi)^3 4\omega}\Big[H^{(2)}_{\nu_-}\left(\frac{p_i}{\omega}e^{-\omega t}\right)H^{(1)}_{\nu_-}\left(\frac{p_i}{\omega}e^{-\omega t}\right) + H^{(2)}_{\nu_+}\left(\frac{p_i}{\omega}e^{-\omega t}\right)H^{(1)}_{\nu_+}\left(\frac{p_i}{\omega}e^{-\omega t}\right)\Big] \tag{110}$$

$$A_{i\to f}=-ie\int\sqrt{-g}\,\bar{U}_{\vec{p}_f,\lambda_f}(x)\gamma_0 A^{\hat{0}}(x)V_{\vec{p}_i,\lambda_i}(x)d^4x \tag{111}$$

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