Niš Lectures on Cosmology 4

Mark Hindmarsh

Department of Physics & Astronomy

University of Sussex

m.b.hindmarsh@sussex.ac.uk

University of Sussex

Problems of the Standard Cosmology

Horizon: CMB temperature $T = 2.728$ K, $\Delta T/T \sim 10^{-5}$. Causal horizon at decoupling $ct_{\rm dec}$ subtends $\simeq 1^\circ$.

Flatness: Friedmann equation: $\Omega - 1 = K/a^2H^2 \propto a$ (a^2) matter (radiation). At e.g. $T = 1$ MeV $\Omega - 1 \simeq 10^{-18}$.

Relics: Extensions of Standard Model contain stable massive particles $m \gg 10^2$ GeV. E.g. GUT monopoles, SUGRA gravitinos.

Fluctuations: How? Why 10^{-5}?

These features of the Universe are understandable in **inflationary cosmology**.

But ... Big Bang initial singularity?

Inflationary Cosmology

Inflation[a] means:

- Early Universe had an accelerating phase $\ddot{a} > 0$
- Huge increase in size: 'number of e-foldings' $N_e \equiv \ln(a_{\rm end}/a_{\rm i}) \simeq 60$
- Quantum fluctuations in a massless scalar field generate perturbations.[b]

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[a] Starobinski 1980; Sato 1981; Guth 1981; Linde 1982; Hawking & Moss 1982; Albrecht & Steinhardt 1982,...

[b] Guth & Pi 1982; Starobinskii 1982; Hawking 1982, Bardeen, Steinhardt & Turner 1983,...

Scalar fields in cosmology

$$S = -\int d^4x \sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi))$$

Recall FRW metric for flat Universe $g_{\mu\nu} = \mathrm{diag}(-1, a^2(t))$

Field equation: $$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) = 0$$

Energy density: $$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{1}{a^2}(\nabla\phi)^2 + V(\phi)$$

Pressure: $$p = \frac{1}{2}\dot{\phi}^2 + \frac{1}{6}\frac{1}{a^2}(\nabla\phi)^2 - V(\phi)$$

Slow roll inflation

Postulate that the scalar field is

- Homogenous: $\phi = \phi(t)$
- Overdamped: $|\ddot{\phi}| \ll 3H|\dot{\phi}|$ ("slow roll")

Sufficient conditions for slow roll:

$$\epsilon = \frac{1}{2} m_{\mathrm{P}}^2 \left(\frac{V'(\phi)}{V(\phi)} \right)^2 \ll 1, \quad |\eta| = \left| m_{\mathrm{P}}^2 \frac{V''(\phi)}{V(\phi)} \right| \ll 1$$

Potential must be "flat" or $\phi \gg m_{\mathrm{P}}$.

Energy density: $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) = V(\phi)(1 + \frac{1}{3}\epsilon)$

Pressure: $p = \frac{1}{2}\dot{\phi}^2 - V(\phi) = -V(\phi)(1 - \frac{1}{3}\epsilon)$

Equation of state: $p \simeq -(1 - \frac{2}{3}\epsilon)\rho$

Solution to Friedmann eqn. $a(t) \propto t^{1/\epsilon}$ ($\to \exp(Ht)$, $H = \sqrt{V/3m_{\mathrm{P}}^2}$).

Amount of expansion

Quantified by 'number of e-foldings' $N_e \equiv \ln(a_{\rm end}/a_{\rm i})$

Integrate

$$3\frac{\dot a}{a}\dot\phi = -V'(\phi), \quad \text{or} \quad \frac{V}{m_{\rm P}^2}\frac{d\phi}{d\ln a} = -V'(\phi)$$

Result:

$$N_e = \ln\left(\frac{a_{\rm end}}{a_{\rm i}}\right) = \frac{1}{m_{\rm P}^2}\int_{\phi_{\rm i}}^{\phi_{\rm end}}\frac{V}{V'}d\phi$$

Example:

$$V = \frac{1}{2}m^2\phi^2 \quad \text{gives} \quad N_e = \frac{1}{2}\frac{(\phi_{\rm end}-\phi_{\rm i})^2}{m_{\rm P}^2}$$

End of inflation

- End of inflation: $\epsilon = 1$ or $|\eta| = 1$
- Example: $V = \frac{1}{2}m^2\phi^2$, $\epsilon = 2m_{\rm P}^2/\phi^2$, giving $\phi_{\rm end} = \sqrt{2}m_{\rm P}$.
- Field oscillates: $\phi \to \phi_0 \sin(mt)/t, \quad a(t) \to t^{2/3}$ ($\mathbf{p} = 0$ bosons).
- Field decays into other species - (p)reheating
- Thermalisation to energy density $\rho_{\rm rh} < V(\phi_{\rm end})$, temperature $T_{\rm rh}$
- **NB** $T_{\rm rh}$ must allow nucleosynthesis (1 MeV)
- **NB** $T_{\rm rh}$ must allow baryogenesis ($T > 100$ GeV)[a]

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[a] or cold electroweak baryogenesis Smit & Tranberg 2004

Solving the horizon problem

Consider mode with comoving momentum $\mathbf{k}$, physical inverse wavenumber $\lambda(t) = a(t)/k$, compared with Hubble length $L_H(t) = H^{-1}$.

Inflation: $a(t) \propto t^{1/\epsilon}$, $H^{-1} = \epsilon t$

Radiation: $a(t) \propto t^{1/2}$, $H^{-1} = 2t$

Matter: $a(t) \propto t^{2/3}$, $H^{-1} = 3t/2$

During inflation the mode's physical wavelength grows faster than the Hubble length.
Let $t_1(k)$ be time at which $\lambda(t) = H^{-1}$ during inflation ("horizon exit")

During standard radiation and matter dominated eras Hubble length grows faster.
Let $t_2(k)$ be time at which $\lambda(t) = H^{-1}$ during standard era ("horizon entry").

Points not now in causal contact were in same Hubble volume during inflation

Sufficient inflation

Modes entering horizon now $\lambda_0(t_0) = a(t_0)/k_0 = H_0^{-1}$.

Require they first crossed horizon (time t_1) during inflation.

Horizon exit for k_0 mode: $a(t_1)k_0^{-1} = H^{-1}(t_1)$.

Hence: $\dfrac{a(t_1)}{a(t_0)} = \dfrac{H_0}{H(t_1)}$

Assume adiabatic expansion between reheat and today:

$$N_e(t_1) = 67 + \ln\left(\frac{T_{\rm rh}}{10^{16}\ \rm GeV}\right) + \frac{1}{6}\ln\frac{g(T_{\rm rh})}{g(T_0)} + \frac{1}{2}\ln\frac{V(t_1)}{V_{\rm end}} + \frac{1}{2}\ln\frac{V_{\rm end}}{\rho_{\rm rh}} - \frac{1}{3}\ln\frac{a_{\rm rh}}{a_{\rm end}}$$

$$\ln(V(t_1)/V_{\rm end}) = 2\epsilon N_e(t_1)$$

Require at least about 60 e-folds of inflation

Solving the flatness problem

Recall Friedmann equation $\Omega - 1 = K/a^2 H^2$

Inflation	$H^2 \propto a^{-2\epsilon}$	$\Omega(t) - 1 \propto a^{-2(1-\epsilon)}$
Reheating/matter	$H^2 \propto a^{-3}$	$\Omega - 1 \propto a$.
Radiation	$H^2 \propto a^{-4}$	$\Omega - 1 \propto a^2$.

$$\frac{\Omega(t_0) - 1}{\Omega(t_1) - 1} \simeq e^{-2N_e(t_1)} \frac{a_{\rm rh}}{a_{\rm end}} \left(\frac{a_{\rm eq}}{a_{\rm rh}}\right)^2 \frac{a_0}{a_{\rm eq}} \sim e^{-10} \left(\frac{a_{\rm rh}}{a_{\rm end}}\right)^{-\frac{4}{3}}$$

Have estimated $\frac{\rho_{\rm rh}}{V_{\rm end}} \sim \left(\frac{a_{\rm rh}}{a_{\rm end}}\right)^{-3}$

Even if inflation begins at t_1 with $\Omega(t_1) \neq 1$ Universe is now very flat.

Disposing of unwanted relics

Universe expands in volume at least $e^{3N_e(t_1)} \sim 10^{87}$ times.

Any unwanted relics must not be created at $T \lesssim T_{\rm rh}$.

E.g. monopoles $T_{\rm rh} < T_{\rm GUT} \sim 10^{16}$ GeV[a]

E.g. gravitinos $T_{\rm rh} < 10^{9}$ GeV[b]

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[a] Zel'dovich & Khlopov 1978, Preskill 1979

[b] Ellis et al 1984; Khlopov & Linde 1984

Fluctuations from scalar field

Field equation: $\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) = 0$

Consider fluctuations around slow-roll: $\phi(x) = \bar{\phi}(t) + \varphi(x)$

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + V''(\bar{\phi})\varphi \simeq 0$$

Introduce conformal time: $a d\tau = dt$, such that $ds^2 = a^2(\tau)(-d\tau^2 + d\mathbf{x}^2)$.

$$\varphi'' + 2\frac{a'}{a}\varphi' - \nabla^2\varphi + 3\eta a^2 H^2 \varphi \simeq 0 \qquad \left(' = \frac{\partial}{\partial\tau}\right)$$

Mode functions

Expand in Fourier modes (assume flat Universe);

$$\varphi(t,\mathbf{x}) = \int \frac{\bar{d}^3k}{2k}\left(a_{\mathbf{k}} f_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_{\mathbf{k}} f^*_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}\right)$$

where functions $f_{\mathbf{k}}$ satisfy

$$f''_{\mathbf{k}} + 2\frac{a'}{a} f'_{\mathbf{k}} + \left(k^2 + 3\eta a^2 H^2\right) f_{\mathbf{k}} = 0$$

Writing $f_{\mathbf{k}} = u_{\mathbf{k}}/a(\tau)$,

$$u''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a} + 3\eta a^2 H^2\right) u_{\mathbf{k}} = 0$$

Zeroth order solution: $f_{\mathbf{k}}(\tau) = \dfrac{k\tau - i}{ak\tau} e^{-ik\tau}$

Density fluctuations

Fluctuations in field $\longrightarrow$ fluctuations in energy density: $\rho(x) \simeq V(\phi(x))$

Hence density contrast $\delta(x) = \dfrac{\rho(x) - \bar{\rho}}{\bar{\rho}} = \dfrac{V'(\bar{\phi})}{\bar{\rho}}\varphi(x)$

Density contrast fluctuation:

$$\langle \delta^2(x) \rangle = \left(\frac{V'(\bar{\phi})}{\bar{\rho}}\right)^2 \langle \varphi^2(x) \rangle = \epsilon \frac{\langle \varphi^2(x) \rangle}{m_P^2}$$

Quantum vacuum fluctuations

Field operator

$$\hat{\varphi}(t,\mathbf{x}) = \int \frac{\text{đ}^3 k}{2k} \left(\hat{a}_{\mathbf{k}} f_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^*_{\mathbf{k}} f^*_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}\right)$$

with $[\hat{a}_{\mathbf{k}}, \hat{a}^*_{\mathbf{k}'}] = 2\omega_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')$ Mean square vacuum[a] fluctuation:

$$\langle 0|\hat{\varphi}^2(x)|0\rangle = \int \frac{\text{đ}^3 k}{2k}|f_{\mathbf{k}}(t)|^2 = \int \frac{dk}{k}\left(\frac{H}{2\pi}\right)^2 (1+k^2\tau^2) = \int \frac{dk}{k}\mathcal{P}_\varphi(k)$$

Inflation happens as $\tau \longrightarrow 0^-$: hence power spectrum $\mathcal{P}_\varphi \longrightarrow \left(\frac{H}{2\pi}\right)^2$ and

$$\mathcal{P}_\delta(k) \to \frac{\epsilon}{m_P^2}\left(\frac{H}{2\pi}\right)^2$$

INFLATION GENERATES SCALE-INVARIANT DENSITY FLUCTUATIONS

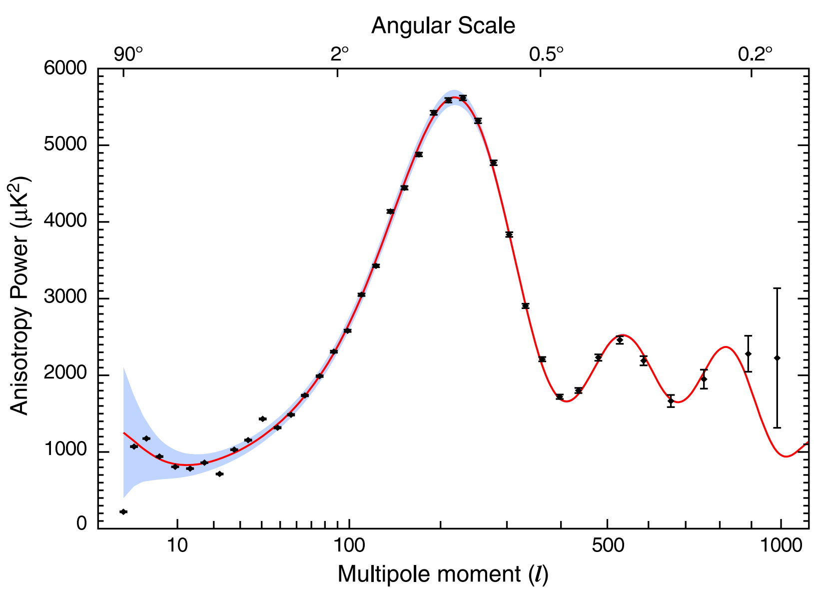
[a] $a_{\mathbf{k}}|0\rangle = 0$ - Bunch-Davies vacuum

From density fluctuations to the CMB

- Density fluctuations $\delta_{\mathbf{k}}$ cause gravitational potential fluctuations $\Phi_{\mathbf{k}}$
- Fluctuations in intensity of CMB radiation arriving here, now through
  - Gravitational redshift - Sachs-Wolf effect
  - Acoustic oscillations - “Doppler peaks”
  - Oscillations are coherent - all started at $t \simeq 0$ with same phase.

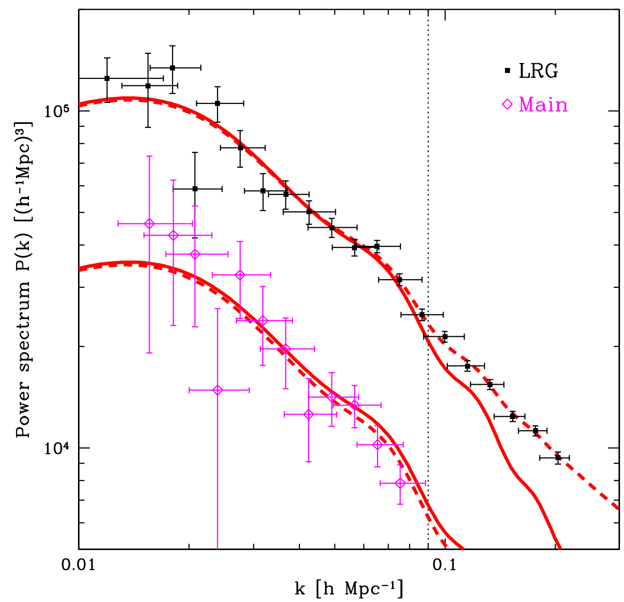
CMB perturbations and large scale structure

- Simple model for CMB angular power spectrum & 3D galaxy power spectrum:
- $\mathcal{P}_\delta = A(k/k_0)^{n_s-1}$ for all species i. (Scalar) spectral index $n_s \simeq 1$.



WMAP 3 angular power spectrum[a]

[a] Hinshaw et al. (2006)



Sloan Digital Sky Survey (2006)[a]

[a] Tegmark et al. (2006), Perceval et al (2007)