Niš Lectures on Cosmology 2

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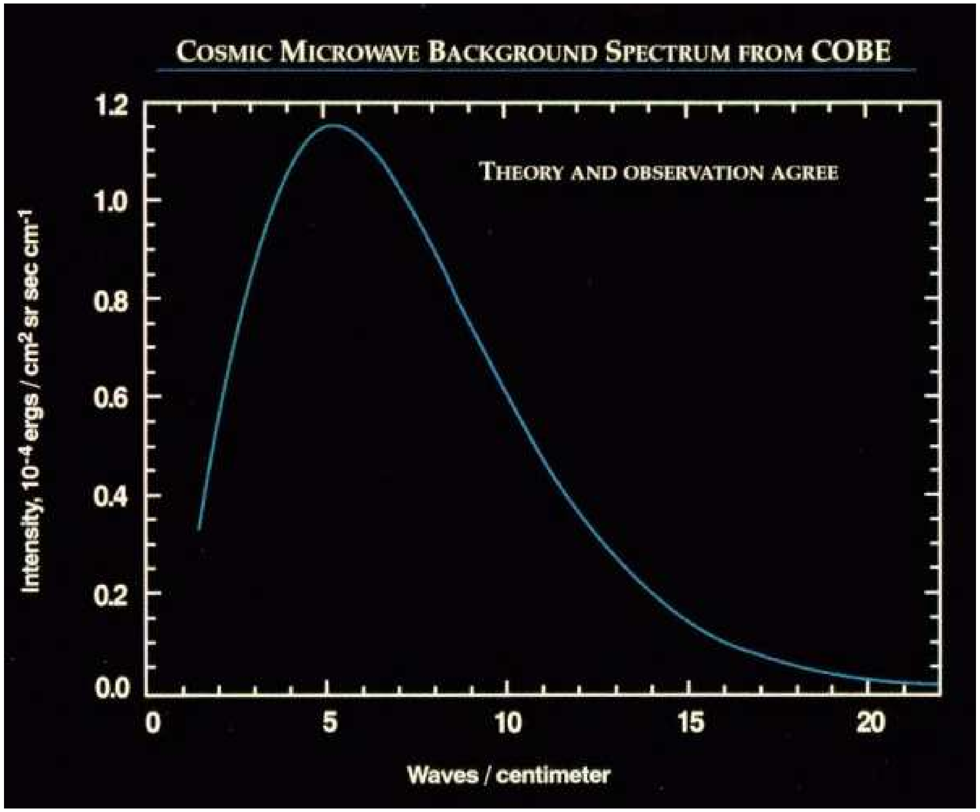
Evidence for thermodynamic equilibrium in the early Universe

- The CMB is a perfect blackbody (as accurately as can be measured)
- The temperature of CMB is $T_{\mathrm{CMB}} = 2.728 \pm 0.004$ K.[a]



[a] Mather et al. (1994)

Thermodynamic equilibrium

When is a species of particle X_i in equilibrium?

Kinetic equilibrium Species can change energy

- Energy density uniform, constant
- State characterised by temperature T

$X_1 + X_2 \leftrightarrow X_1 + X_2$ gives $T_1 = T_2$. $\boxed{E = -\frac{d}{d\beta} \ln Z(\beta, \mu)}$

Chemical equilibrium Species can change number

- Number density uniform, constant
- State characterised by chemical potential μ

$X_1 + X_2 \leftrightarrow X_3 + X_4$ gives $\mu_1 + \mu_2 = \mu_3 + \mu_4$. $\boxed{N = \frac{1}{\beta} \frac{d}{d\mu} \ln Z(\beta, \mu)}$

Conditions for and consequences of equilibrium

Particle reaction rates large compared with expansion rate $H \propto 1/t$

$$n\langle \sigma v\rangle \gg H \qquad \begin{cases} \sigma & \text{Scattering cross-section} \\ n & \text{Number density of scatterers} \\ v & \text{Relative speed} \\ \langle \ldots \rangle & \text{Thermal average} \end{cases}$$

Early Universe very close to thermodynamic equilibrium: expansion isentropic.

$S = sa^3 =$ const. Entropy density s.

Thermodynamic relations:

$$s = \frac{dp}{dT}, \quad sT = \rho + p \qquad \left(\to \rho = T^2 \frac{d}{dT}\left(\frac{p}{T}\right)\right)$$

NB Equilibrium fails for neutrinos at $T \simeq 1$ MeV, WIMPs at $T \simeq 1 - 10$ GeV.

NB Very far from equilibrium in most parts of Universe now.

Ideal gases in equilibrium

Eqm. distribution function:

Particle energy $E_{\mathbf{k}}(t) = \sqrt{(\mathbf{k}^2(t) + m^2)}$, degeneracy g (spin, colour, ...)

$$f(\mathbf{k}, t) = \frac{g}{[e^{(E_{\mathbf{k}}-\mu)/T} + \eta]}$$

Fermi-Dirac: $\eta = +1$, Bose-Einstein $\eta = -1$

Number density	$n = \int đ^3k\, f(\mathbf{k}, t)$	$= \frac{g}{2\pi^2} \int_m^\infty dE \frac{(E^2-m^2)^{\frac{1}{2}} E}{e^{(E-\mu)/T}+\eta}$
Energy density	$\rho = \int đ^3k\, f(\mathbf{k}, t) E_{\mathbf{k}}$	$= \frac{g}{2\pi^2} \int_m^\infty dE \frac{(E^2-m^2)^{\frac{1}{2}} E^2}{e^{(E-\mu)/T}+\eta}$
Pressure	$p = \int đ^3k\, f(\mathbf{k}, t) \mathbf{k}^2/3E_{\mathbf{k}}$	$= \frac{g}{2\pi^2} \int_m^\infty dE \frac{(E^2-m^2)^{\frac{3}{2}}}{e^{(E-\mu)/T}+\eta}$

Relativistic and non-relativistic limits ($\mu = 0$)

Quantity	*Relativistic B*	$\times$ *(F)*	*Non-relativistic* $(T \ll m)$
Number density	$n_r = g\frac{\zeta(3)}{\pi^2}T^3$	$\left(\frac{3}{4}\right)$	$n_m = g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}\exp(-m/T)$
Energy density	$\rho_r = g\frac{\pi^2}{30}T^4$	$\left(\frac{7}{8}\right)$	$\rho_m = mn_m(T)$
Pressure	$p_r = g\frac{\pi^2}{90}T^4$	$\left(\frac{7}{8}\right)$	$p_m = n_m(T)T \ll \rho_m$
Entropy density	$s_r = g\frac{2\pi^2}{45}T^3$	$\left(\frac{7}{8}\right)$	$s_m = mn_m(T)/T \ll s_r(T)$

NB Isentropic expansion $s_r \propto a^{-3}$ means $T_r \propto 1/a$.

NB $m = 0$ particles out of chemical equilibrium maintain distribution: $E \propto 1/a$.

NB Equations of state: $p_r = \rho_r/3$, $p_m \simeq 0$

Effective numbers of degrees of freedom

Many species at the same T: $\rho = \sum_{B,F}\left(g_B + \frac{7}{8}g_F\right)\frac{\pi^2}{30}T^4$

Different $T_{B,F}$: $\rho = \sum_{B,F}\left(g_B\left(\frac{T_B}{T_\gamma}\right)^4 + \frac{7}{8}g_F\left(\frac{T_F}{T_\gamma}\right)^4\right)\frac{\pi^2}{30}T_\gamma^4$

More generally, define effective number of degrees of freedom relative to ideal gas of massless bosons. Let $x_i = m_i/T$

Energy density $g_i(T) = \frac{30}{\pi^2 T_\gamma^4}\rho_i(T) \quad = \frac{15}{\pi^4}g_i x_i^4 \int_1^\infty \frac{(y^2-1)^{\frac{1}{2}}}{e^{x_i y}+\eta_i} y\,dy$

Entropy density $h_i(T) = \frac{45}{2\pi^2 T_\gamma^3}s_i(T) \quad = \frac{45}{4\pi^4}g_i x_i^3 \int_1^\infty \frac{(y^2-1)^{\frac{1}{2}}}{e^{x_i y}+\eta_i}\frac{4y^2-1}{3y}dy$

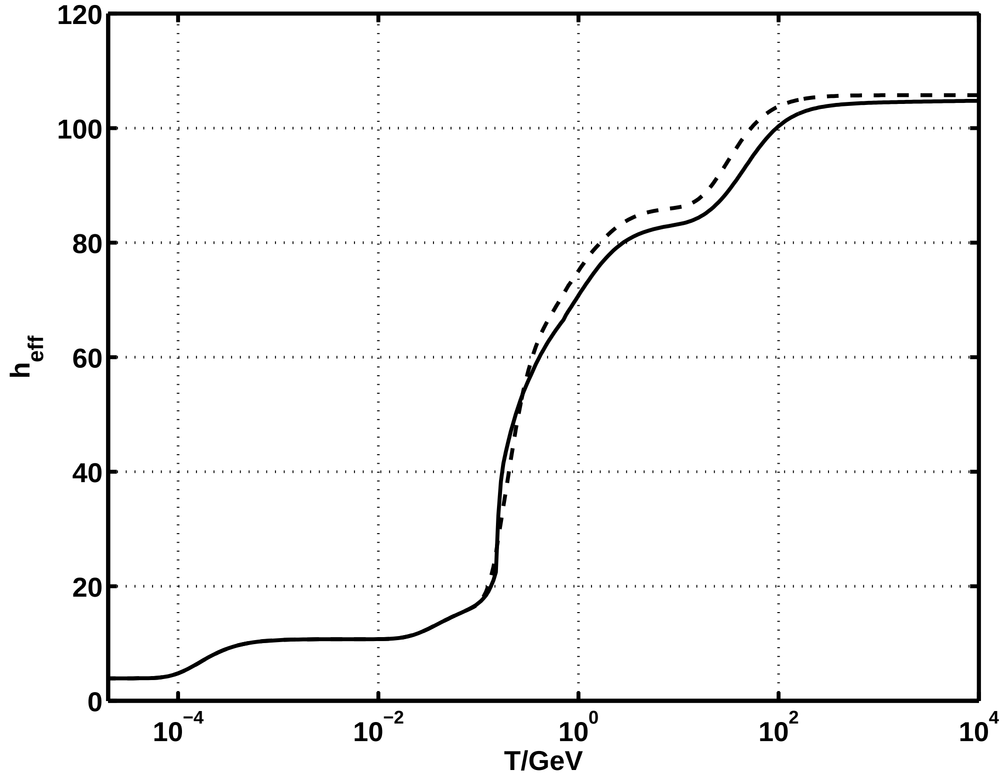
Total: $g_{\text{eff}}(T) = \sum_i g_i(T), \quad h_{\text{eff}}(T) = \sum_i h_i(T)$

Standard Model degrees of freedom: mostly coloured at high T

	γ	0	2	g	0	16	
	ν_e	$\lesssim 1$ eV	2	u	3 MeV	12	
	ν_μ	$\lesssim 1$ eV	2	d	7 MeV	12	
	ν_τ	$\lesssim 1$ eV	2	s	76 MeV	12	
	e	0.5 MeV	4	c	1.2 GeV	12	
	μ	106 MeV	4	b	4.2 GeV	12	
	τ	1.7 GeV	4	t	174 GeV	12	
	W	80 GeV	6				
	Z	91 GeV	3				
	h	$>$ 115 GeV	1				
$\gg$ **174 GeV:**			$\frac{7}{8}18 + 12$			$\frac{7}{8}72 + 16$	79/106.75
10^{-4} **GeV:**			$h_\nu(T) + 2$			0	0/3.92

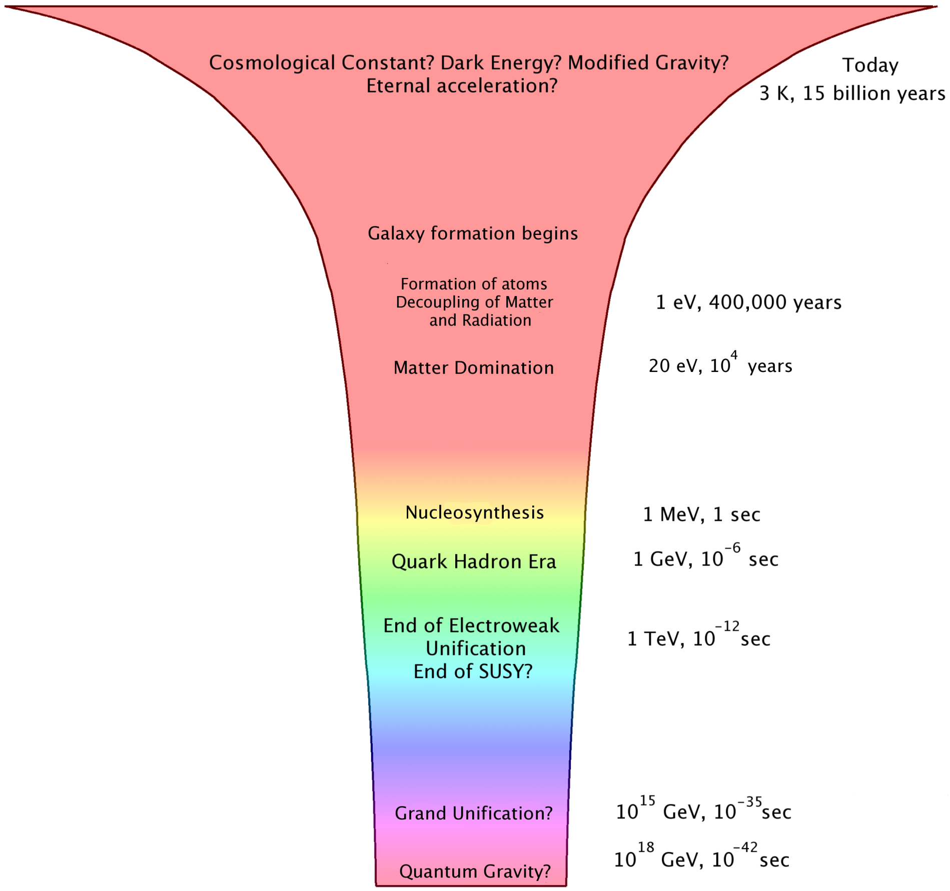
SM Degrees of freedom (including QCD interactions)

Olive 1981 (dashed), Hindmarsh and Philipsen 2005 (solid)



Temp.	Event
100 GeV	t non-relativistic
1 GeV	b non-relativistic
500 GeV	c, τ non-relativistic
200 MeV	QCD phase transition
30 MeV	μ non-relativistic
2 MeV	ν freeze-out
0.2 MeV	e non-relativistic
1 eV	matter = radiation
0.1 eV	photon decoupling

Thermal history of the Universe



Decoupling and Freeze-out

Decoupling: departure from kinetic equilibrium

Freeze-out: departure from chemical equilibrium

Estimate decoupling/freeze-out temperature: $n\langle\sigma v\rangle \simeq H$

H from Friedmann equation (relativistic gas): $H^2 = \frac{1}{3m_P^2} g_{\rm eff} \frac{\pi^2}{30} T^4$

$$n\langle\sigma v\rangle \sim \sqrt{g_{\rm eff}}\frac{T^2}{m_P}$$

Example: Neutrino decoupling

$$n\langle\sigma v\rangle \sim \sqrt{g_{\rm eff}}\frac{T^2}{m_P}$$

Kinetic equilibrium maintained by: $\nu e \leftrightarrow \nu e$ •

Cross-section: $\sigma \sim G_F^2 T^2$, $v \simeq 1$, $n_e \sim T^3$

Decoupling temperature $T_{d,\nu}^3 \sim g_{\rm eff}^{1/2}/m_P G_F^2$

Gives $T_{d,\nu} \sim 1{\rm MeV}$

Hot dark matter ($T_{d,\nu} \gg m_\nu$)

Note: $E_k(t) = |\mathbf{k}|(a_0/a(t))$, so distribution function

$$f(\mathbf{k},t) = \frac{g}{[e^{|\mathbf{k}|a_0/Ta(t)} + \eta]}$$

Decoupled relativistic species has thermal distribution with $T(t) = T_0 a_0/a(t)$

Example: WIMP freeze-out

$$n\langle\sigma v\rangle \sim \sqrt{g_{\rm eff}}\frac{T^2}{m_P}$$

Chemical equilibrium maintained by (e.g.): $XX \leftrightarrow q\bar{q}$ •

Cross-section: $\langle\sigma v\rangle \sim G_F^2 m_q^2$, $n_X \sim (mT)^{3/2}e^{-m/T}$

Freeze-out temperature $(m/T)^{1/2}e^{-m/T} \sim g_{\rm eff}^{1/2}/m_P m G_F^2 m_q^2$

Gives $T_{f,X} \simeq m_X/25$

Cold dark matter ($T_{f,X} \ll m_X$)

Decoupled non-relativistic species does not have a thermal distribution

Relic density

Non-relativistic - **“cold dark matter”**:

Number density at freeze-out:

$$n_X \simeq g_{\rm eff}^{1/2}(m_X T_f)^{3/2} \exp(-m_X/T_f) \sim g_{\rm eff}^{1/2} T_f^2/m_P G_F^2 m_q^2$$

Number density by diluted by expansion: $n_X \propto a^{-3}$ - same as entropy s

$$\Omega_X = \left.\frac{m_X n_X}{\rho_c}\right|_{T_0} \frac{s_0}{s} \sim \frac{g_{\rm eff}^{1/2}}{h_{\rm eff}} \frac{T_0^3}{\rho_c} \frac{1}{m_P G_F^2 m_q^2} \frac{m_X}{T_f}$$

Result (after a more careful calculation):

$$\Omega_X h^2 \simeq 0.06(m_X/T_f) \sim 0.15 \qquad (\Omega_X \sim 0.3)$$

NB Logarithmically dependent on mass, cross-section of WIMP.

NB Depends on $g_{\rm eff}^{1/2}(T_f)/h_{\rm eff}(T_f)$.

More accurate WIMP relic density calculation

Mass m, number density n, annihilations $XX \longrightarrow \ldots$ with total cross-section σ.

Boltzmann equation (assuming non-relativistic particles): [a]

$$\dot{n} + 3\frac{\dot{a}}{a}n = -\left\langle \sigma v_{\mathrm{Møl}} \right\rangle \left(n^2 - n_{\mathrm{eq}}^2\right)$$

Let $x = m/T$, $Y = n/s$:

$$\frac{dY}{dx} = \frac{1}{3H}\frac{ds}{dx}\left\langle \sigma v_{\mathrm{Møl}} \right\rangle \left(Y^2 - Y_{\mathrm{eq}}^2\right).$$

Friedmann $H^2 = 8\pi\rho/3M_{\mathrm{P}}^2$, define $g_*^{1/2}(T) = \frac{h_{\mathrm{eff}}}{g_{\mathrm{eff}}^{1/2}}\left(1 + \frac{T}{3}\frac{d\ln h_{\mathrm{eff}}}{dT}\right)$.

$$\frac{dY}{dx} = \left(\frac{\pi}{45}\right)^{\frac{1}{2}} g_*^{\frac{1}{2}}(T)(mM_{\mathrm{P}})\left\langle \sigma v_{\mathrm{Møl}} \right\rangle \left(Y^2 - Y_{\mathrm{eq}}^2\right)\frac{1}{x^2}$$

[a] Møller velocity: $v_{\mathrm{Møl}} = \left((\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2\right)^{\frac{1}{2}}$ (Gondolo & Gelmini 1991)

Approximate solution:

$$Y_0 \simeq \left(\frac{45}{\pi}\right)^{\frac{1}{2}} \frac{1}{m M_{\mathrm{P}} \langle \sigma v_{\mathrm{M\o l}} \rangle_{T_f}} \frac{x_f}{g_*^{\frac{1}{2}}(T)}$$

Requires m, $\langle \sigma v_{\mathrm{M\o l}} \rangle_T$ (Particle physics) $g_*^{\frac{1}{2}}(T)$ (cosmology).

Let $\langle \sigma v_{\mathrm{M\o l}} \rangle = a_{(0)} + a_{(1)}/x + \cdots$ ($1/x = T/m$, kinetic energy per unit mass)

$$\Omega_X h^2 \simeq \frac{10^{-10}\ \ \mathrm{GeV}^{-2}}{a_{(0)}} \frac{x_f}{g_*^{\frac{1}{2}}(T)}$$

Estimate $a_{(0)} \sim (1/3) \times 10^{-9}$ GeV^{-2} ($XX \longrightarrow b\bar{b}$), $x_f \simeq 25$, $g_*^{\frac{1}{2}} \simeq 5$.

$$\Omega_X h^2 \sim 10^{-1}$$

Even more accurate: computer packages for SUSY relics

- Leading candidate for a WIMP in Supersymmetric Standard Model: Neutralino.
- Neutralino χ is a mixture of $\tilde{\gamma}$, $\tilde{Z}$, $\tilde{h}$ and $\tilde{H}$ (spin-$\frac{1}{2}$).
- Very carefully studied in Minimal Supersymmetric Standard Model
- Collective wisdom implemented in computer packages:

DarkSUSY[a] **MicrOMEGAs**[b]

- New equation of state[c]

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[a] Gondolo et al 2004
[b] Bélanger et al 2001, 2004
[c] Hindmarsh & Philipsen (2005)

What else could dark matter be?

- Not baryonic ($\Omega_b \simeq 0.05$) from nucleosynthesis
- Axion, gravitino, axino, ... see Laura Covi's talk at COSMO 07 (http://www.cosmo07.info/)

Conventions

- Natural Units: $\hbar = 1$, $c = 1$, $k_B = 1$
- Natural Unit converter:

Quantity	Nat. U.	S.I. Conversion	
Energy:	GeV	1.6022×10^{-10}	Joule
Temperature:	GeV	1.1605×10^{13}	K
Mass:	GeV	1.7827×10^{-27}	kg
Length:	GeV^{-1}	1.9733×10^{-16}	m
Time:	GeV^{-1}	6.5822×10^{-25}	s

- Planck Mass (Energy): $M_P = \sqrt{\hbar c^5/G} = 1.2211 \times 10^{19}$ GeV
- Reduced Planck Mass $m_P = \sqrt{\hbar c^5/8\pi G} = 2.436 \times 10^{18}$ GeV