Induced Schwinger Processes

Semiclassical Treatment

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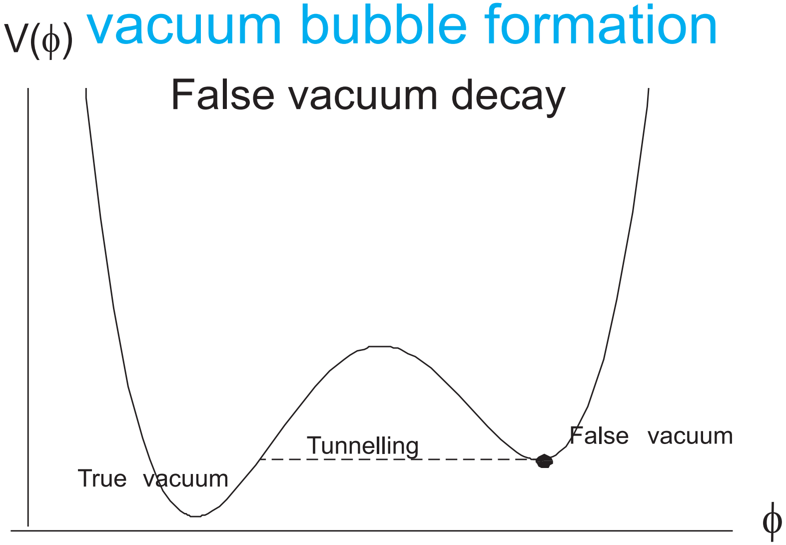
Outline of the talk

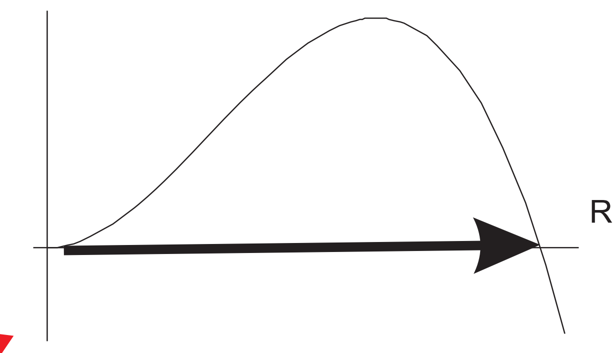
- Intro: Vacuum Decay and Schwinger Processes
- Decay of a Monopole: the Leading Exponential
  - Feynman Path Integral
  - Dominating Classical Configurations
- Decay of a Monopole: Sub-Leading Prefactor
  - Green Functions in an External Field
  - Saddle-Point Approximation for Semiclassical Analysis
- Meson Decay at Zero Temperature
  - Schwinger Process for Thirring Model
- Thermal Corrections to Meson Decay

Vacuum Decay as Tunnelling

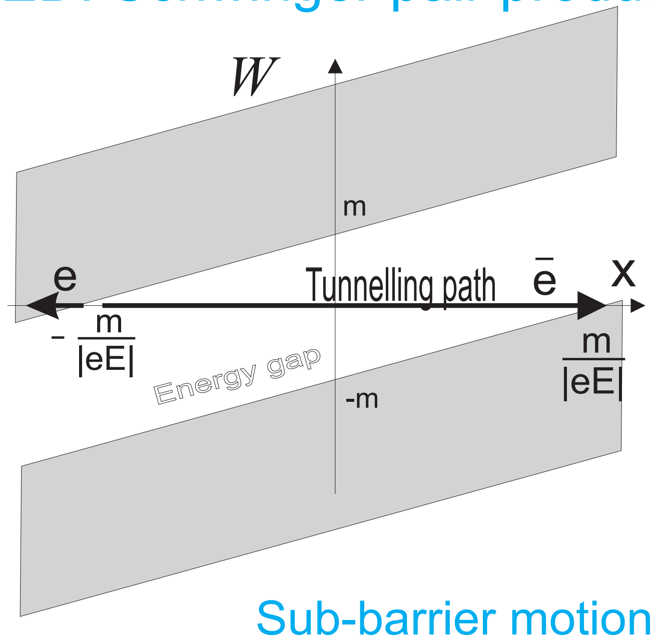
Deformed potential

Scalar Field Theory:
vacuum bubble formation

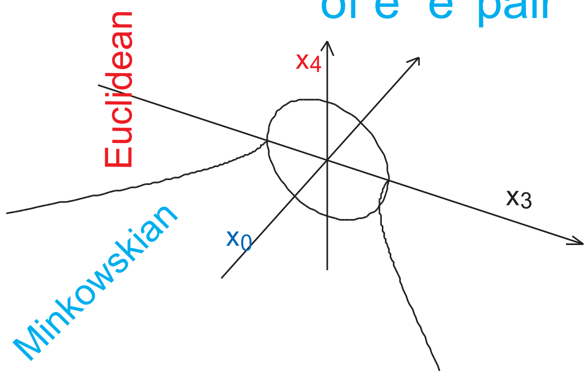




QED: Schwinger pair production



Sub-barrier motion
of $e^+ e^-$ pair



External field

Different reasons for same physics

QED Reminder: Spontaneous Schwinger

The imaginary part of the effective Euler–Heisenberg–Schwinger Lagrangian describes probability w of e^+e^- pair production from vacuum

$$w = 2\,\mathrm{Im}L_{eff} \sim \mathrm{Im}\int \frac{ds}{s^2} e^{im^2 s}\left(\frac{eE}{\sinh(eEs)} - \frac{1}{s}\right) \sim$$
$$\sim \sum_{n=0}^{\infty} \frac{1}{n^2} e^{-\frac{\pi m^2 n}{eE}}$$

[Euler,Heisenberg 1935; Schwinger 1951]

This expression already has the following characteristic features:

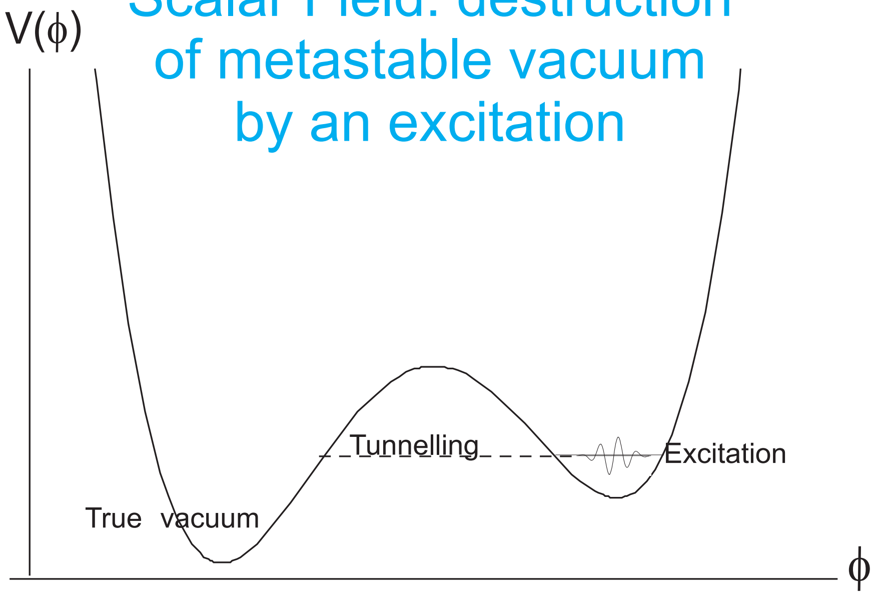
- Non-perturbative behaviour in E
- Finite imaginary part is extracted directly from the Schwinger proper-time integral
- Semiclassical interpretation is easy: in the above sum, n can be thought of as world-line instanton topological number.

These properties will manifest themselves in a more complicated fashion in what we do below for induced Schwinger phenomena.

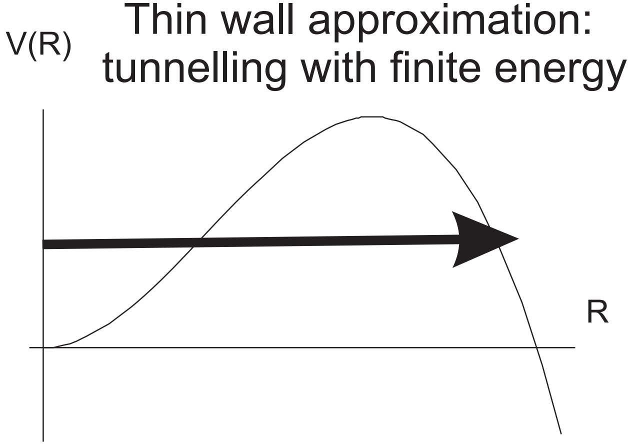
Spontaneous vs. Induced

Deformed potential

Scalar Field: destruction
of metastable vacuum
by an excitation

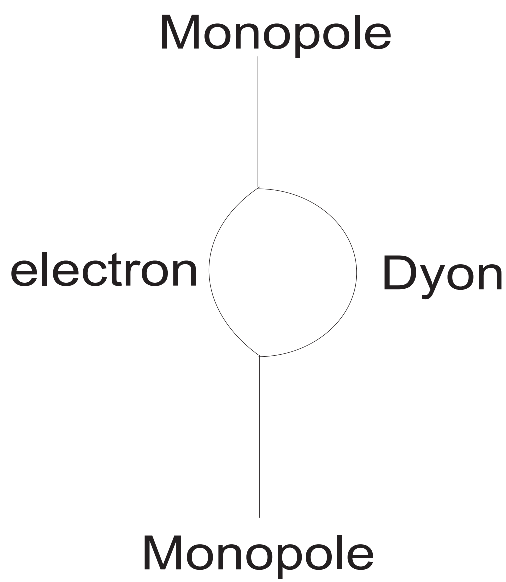


Thin wall approximation:
tunnelling with finite energy



External field

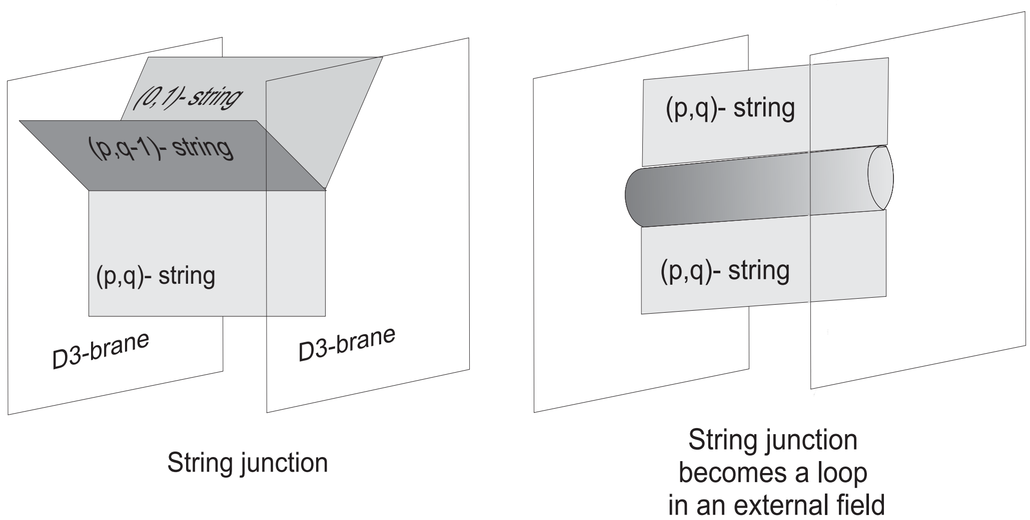
Forbidden process
in a gauge theory with fermions
becomes non-perturbatively allowed



The similarity is not literal as in the previous case
but still the physics is essentially close

String Theory Motivation

World-sheets of electrically and magnetically charged (p, q)-strings may form a vertex shown below



Monopole in an SU(2) theory can be thought as an D-string stretched between two D3-branes. String theory provides us with a junction, which can account for the decay of BPS states in low-energy theory. The junction allows for a loop when an external field is on.

String Theory, Thin Wall Approximation and Electrodynamics

More generally, the action of a compact p-brane configuration is given by a sum of area term and volume term

$$S = S_{area} + S_{volume} = T\int_{area} - Q\int_{volume} \Phi$$

where T is p-brane tension, $G_{\mu\nu}$ is the metric, induced by brane embedding into target-space, Q — brane charge, Φ — flux density of the external $(p+2)$-form field **[Gorsky 2001]**.

This formula is a natural generalization of electrodynamics 1-particle action

$$S = S_{area} + S_{volume} = \int m ds + e\int A_\mu dx^\mu$$

On the other hand, this is the action for a false vacuum bubble in thin wall approximation **[Voloshin 1985]** in 1+1 dimensions

$$S = S_{area} + S_{volume} = \int (\mu\sqrt{\dot{\rho}^2 + \rho^2} - \frac{1}{2}\epsilon\rho^2)$$

where μ is the action density per unit of bubble boundary, ϵ is the parameter, proportional to energy difference between the two vacua.

Some History - 1

- **[Popov 1972]** “Pair production in a variable and uniform field ...”. Imaginary time formalism introduced.
- **[Stone 1976]**, “Semiclassical Methods For Unstable States”. Scalar field vacuum decay treated semiclassically.
- **[Affleck, De Luccia 1979]**, “Induced Vacuum Decay”. Semiclassical treatment expanded to induced processes.
- **[Agaev et. al. 1984]** “Quasiclassical Description Of The Vacuum Instability In An External Nonabelian Gauge Field”. 1-particle formalism applied to spontaneous Schwinger processes.
- **[G. V. Dunne and C. Schubert, 2005]** “Worldline instantons and pair production in inhomogeneous fields”. One-particle method systematically developed for both leading exponential and the pre-exponential factor.

Some History - 2

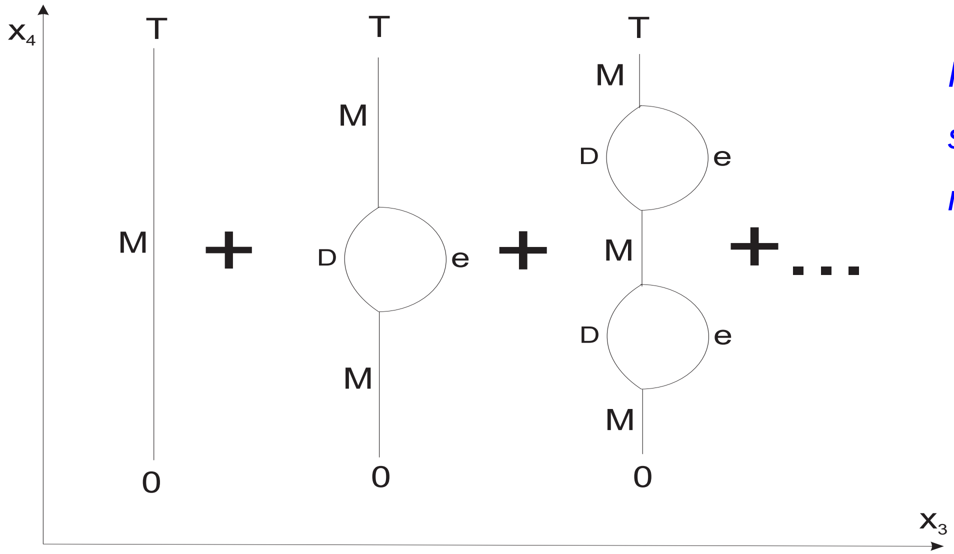
Monopoles are perturbatively stable. However, in an unstable vacuum background they may catalyze vacuum decay, which is interpreted as decay of BPS state itself. This instability may be caused by deforming the potential or by an introduction of an external field.

- **[Steinhardt 1981]** “Monopole And Vortex Dissociation And Decay Of The False Vacuum.” Monopole in the context of scalar field deformed vacuum.
- **[Gorsky 2001]** “Schwinger type processes via branes and their gravity duals.” BPS decay in an external field suggested form semiclassical string paradigm.
- **[Dymarsky, Melnikov 2003]** “Comments on BPS bound state decay.” Marginal stability curve for “monopole + fermion” bound state studied quasiclassically.
- **[Monin 2005]** “Monopole decay in the external electric field.” Leading exponential factor calculated semiclassically.

Semiclassical Path Integral

Semiclassical approximation:

- Find closed-loop trajectories in Euclidean time
- Calculate fluctuation determinants



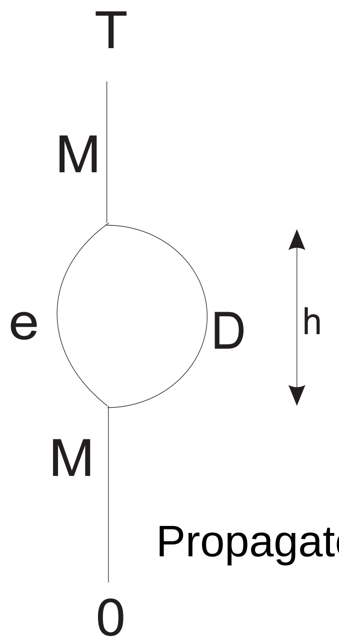
Full 1-loop Green function of a monopole is obtained by summing over all the insertions of electron-dyon loop into monopole's Euclidean trajectory.

$$G(T,0) = G^{(0)}(T,0) + G^{(1)}(T,0) + G^{(2)}(T,0) + \cdots$$

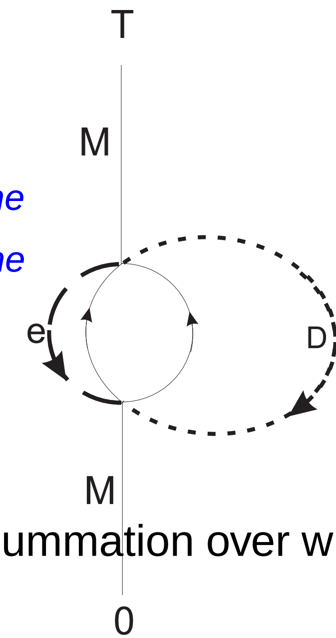
Worldline Instantons

First correction to (scalar) Green function

$$G^{(1)}(T,0) = \int \mathcal{D}y e^{-M_m \int \sqrt{\dot{y}^2} d\tau} \mathcal{D}x \mathcal{D}z e^{-S[x,z,A]}$$



Electron and dyon can go round the loop multiply, winding over it with some respective winding numbers m, n.



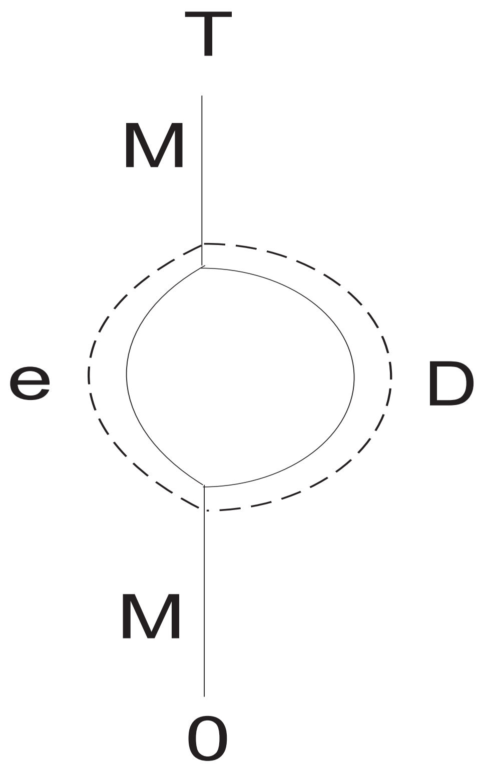
*One-loop Euclidean configuration with multiply winding trajectories (**"world-line instantons"**). Dashed bold arcs correspond to extra winding paths, arrows indicate winding direction.*

Propagator correction becomes after resummation over winding number m, n

$$G^{(1)}_{resummed}(x,z) \sim \sum_{m,n} K_{m,n} e^{-S_{cl}^{(m,n)}}$$

Negative Mode

Dilatation of the electron-dyon loop



Fluctuation determinants yield us the first corrections to semiclassical exponential decay factors. The special fluctuation, corresponding to the overall dilatation of the loop, possesses a negative eigenvalue. This negative eigenmode is the source of the imaginary part of the mass correction.

Exponential and Preexponential

The first correction due to electron-dyon loop is

$$G^{(1)}(T,0) = \int \mathcal{D}y e^{-M_m \int \sqrt{\dot{y}^2} d\tau} \mathcal{D}x \mathcal{D}z e^{-S[x,z,A]},$$

$S[x,z,A]$ – action for the particles in the external field A_μ, x and z electron and dyon coordinates,

$$S[x,z,A] = m \int \sqrt{\dot{x}^2} du + ie \int A^{ext}(x) \dot{x} du + M_d \int \sqrt{\dot{z}^2} dv - ie \int A^{ext}(z) \dot{z} dv \ .$$

Semiclassically,

$$G^{(1)}(T,0) = \int d^4 y G^{(0)}(x,y) G^{(0)}(y+\Delta y, z) K e^{-S_{cl}},$$

where S_{cl} is the classical action of dyon and electron; K contains contributions from the Jacobian and from non-zero modes.

On the other hand, $\delta G(x,z) = -\delta m^2 \int d^4 y G^{(0)}(x,y) G^{(0)}(y+\Delta y, z),$ thus

$$\delta m^2 = K e^{-S_{cl}}$$

Exponential factor

Notations: μ_1, μ_2, m masses of electron, dyon and monopole respectively.

The equations of motion are

$$m \frac{d}{du} \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} = -ieF_{\mu\nu}(x)\dot{x}_\nu$$

We consider $\vec{E} = (0, 0, E)$, hence Euclidean trajectories of the sub-barrier particles (electron and dyon) are just the arcs of circle of angular size θ_1, θ_2

$$\theta_1 = \cos^{-1} \frac{m^2 + \mu_1^2 - \mu_2^2}{2m\mu_1}$$
$$\theta_2 = \cos^{-1} \frac{m^2 - \mu_1^2 + \mu_2^2}{2m\mu_2}$$

The leading semiclassical exponential term in Γ becomes

$$\Gamma \sim e^{-\left(\frac{m_e^2}{eE}\theta_1 + \frac{M_d^2}{eE}\theta_2 - \frac{m_e M_d}{eE}\sin(\theta_1 + \theta_2)\right)}$$

Unfortunately, the prefactor is not easily recovered within this method

Second Quantized Calculation

Advantage of second-quantized approach – 1-loop preexponential obtained at the same price

Fermionic Green function of a dyon in the constant external field:

$$G(x,x') = \frac{1}{16\pi^2}\int ds \frac{eE}{\sinh(eEs)}\frac{gE}{\sin(gEs)} e^{im^2 s + i\frac{gE(x-x')^2_\perp}{4\tan(gEs)} + i\frac{eE}{4\tanh(eEs)}(x-x')^2_\parallel} \times$$

$$\times e^{-i\frac{1}{2}eE(x_0+x'_0)(x_3-x'_3) - i\frac{1}{2}gE(x_1+x'_1)(x_2-x'_2) + i\sigma_{03}eE + i\sigma_{12}gE} \times$$

$$\times \left\{ m - \frac{gE\gamma_\perp(x-x')_\perp}{2\tan(gEs)} - \frac{eE\gamma_\parallel(x-x')_\parallel}{2\tanh(eEs)} + \right.$$

$$\left. + \frac{\gamma^0 eE}{2}(x_3 - x'_3) - \frac{\gamma^3 eE}{2}(x_0 - x'_0) + \frac{\gamma^1 gE}{2}(x_2 - x'_2) - \frac{\gamma^2 gE}{2}(x_1 - x'_1) \right\}$$

$\parallel$ denotes directions $0,3$, $\perp$ directions $1,2$, electric field is constant and directed along axis 3, summation over respectively repeating $\parallel$ and $\perp$ is supposed.

Disadvantage – calculation requires knowledge of an exact

Green function, available for a limited class of fields.

Loop Correction

A correction to the Green function due to the electron-dyon loop is

$$\delta G(t,0) = \int G_M(t,x)\mathrm{tr}\left[G(x,y)_E^{(Ext)} G(y,x)_D^{(Ext)}\right] G_M(x,0)dxdy,$$

indices M, E, D denoting monopole, electron and dyon correspondingly. After calculating the trace, the correction to monopole Green function becomes

$$\delta G_m(T,0) = \frac{1}{2^{18}\pi^4}\lambda^2 e g^3 E^4 \int \frac{d\alpha_1\, d\alpha_2 d\alpha_3\, d\alpha_4\, dz\, dw\, \mathrm{e}^{-(B+S_\parallel+S_\perp)}}{\alpha_1 \sin\alpha_1 \sin\alpha_2 \sinh(\frac{g}{e}\alpha_2)\, \alpha_3\, \alpha_4 \sinh(\frac{g}{e}\alpha_4)\sinh(\frac{g}{e}\alpha_3)} \times$$

$$\times\left(m_e M_d \cosh(\frac{g}{e}\alpha_2)\cos(\alpha_1-\alpha_2) + \left(\frac{eE}{2}\right)^2 (w-z)_\parallel^2 \frac{\cosh(\frac{g}{e}\alpha_2)}{\sin\alpha_1\ \sin\alpha_2} + \frac{egE^2}{4}(w-z)_\perp^2 \frac{\cos(\alpha_1-\alpha_2)}{\sinh(\frac{g}{e}\alpha_2)}\right)$$

where

$$\begin{aligned}
B &= \frac{m_e^2}{eE}\alpha_1 + \frac{M_d^2}{eE}\alpha_2 + \frac{M_m^2}{eE}(\alpha_3+\alpha_4) \\
S_\parallel &= \frac{eE}{4\alpha_4} z_\parallel^2 + \frac{eE}{4s}(T-w)_\parallel^2 + \frac{eE}{4}(w-z)_\parallel^2(\cot\alpha_1 + \cot\alpha_2) \\
S_\perp &= \frac{gE}{4} z_\perp^2 \coth(\frac{g}{e}\alpha_4) + \frac{gE}{4} w_\perp^2 \coth(\frac{g}{e}\alpha_3) + \frac{eE}{4}\frac{(w-z)_\perp^2}{\alpha_1} + \frac{eE}{4}(w-z)_\perp^2 \coth(\frac{g}{e}\alpha_2) - \\
&\quad - i\frac{gE}{2}(w_1 z_2 - w_2 z_1).
\end{aligned}$$

Loop Correction

Integrating out z and w and introducing Feynman variables $\alpha_3 = Ax,\ \alpha_4 = A(1-x)$ one gets

$$\delta G(T) \sim \frac{\lambda^2 g^2}{e} \int \frac{d\alpha_1\, d\alpha_2\, A\, dA}{\alpha_1\ \sin\alpha_1\ \sin\alpha_2\ \sinh(\frac{g}{e}\alpha_2)}\, \mathrm{e}^{-\left[\frac{m_e^2}{eE}\alpha_1 + \frac{M_d^2}{eE}\alpha_2 + \frac{M_m^2}{eE}A + \frac{\frac{eE}{4}T^2}{A + \frac{\sin\alpha_1\ \sin}{\sin(\alpha_1+\alpha_2}}\right.}$$

$$\times \frac{1}{\left[(\frac{e}{\alpha_1} + g\cot\frac{g\alpha_2}{e})\sinh\frac{gA}{e} + g\cosh\frac{gA}{e}\right]\left[A(\cot\alpha_1 + \cot\alpha_2) + 1\right]} \times$$

$$\times \left\{ m_e M_d \cosh(\frac{g}{e}\alpha_2)\cos(\alpha_1 - \alpha_2) + eE\frac{\cosh(\frac{g}{e}\alpha_2)A}{\sin\alpha_1\ \sin\alpha_2[A(\cot\alpha_1 + \cot\alpha_2) + \cdots} \right.$$

$$+ \left(\frac{eET}{2}\right)^2 \frac{\cosh(\frac{g}{e}\alpha_2)}{\sin\alpha_1\ \sin\alpha_2[A(\cot\alpha_1 + \cot\alpha_2) + 1]^2} +$$

$$\left. + egE\frac{\cos(\alpha_1 - \alpha_2)\ \sinh(\frac{g}{e}A)}{\alpha_1 \sinh(\frac{g}{e}\alpha_2)\left[\left(\frac{e}{\alpha_1} + g\cot\frac{g\alpha_2}{e}\right)\sinh\frac{gA}{e} + g\cosh\frac{gA}{e}\right)\right]} \right\}.$$

Further we shall have to evaluate this via saddle-point method.

Saddle-Point Integral 1

The function to be minimized $\nu f(A) = -\frac{M_m^2}{eE}\left[A + \frac{(eE)^2}{4M_m^2}T^2\frac{1}{A+const}\right]$ satisfies the condition of saddle-point method applicability. Thus

$A_0 = \frac{eET}{2M_m} - \frac{\sin\alpha_1 \sin\alpha_2}{\sin(\alpha_1+\alpha_2)}$, and the second derivative is $\frac{\partial^2 f}{\partial A^2} = \frac{4M_m^3}{(eE)^2 T}$.

Euclidean propagator of a scalar particle in an external field is $G_m(T,0) = \frac{1}{16\pi^{3/2}}\frac{gE}{\sqrt{M_m T}}\frac{\mathrm{e}^{-M_m T}}{\sinh\frac{gET}{2M_m}}$, and the leading-order contribution to its variation

$$\delta G_m(T,0) = -\frac{1}{8\sqrt{2}\pi^{3/2}}\delta M_m\, gE\sqrt{\frac{T}{M_m}}\,\frac{\mathrm{e}^{-M_m T}}{\sinh\frac{gET}{2M_m}}.$$

Comparing the two expressions for $\delta G(T,0)$ one gets

$$\mathrm{Im}\,\delta M_m = -\frac{1}{2^7\sqrt{2}\pi^{3/2}}\frac{\lambda^2 g}{M}\int\frac{d\alpha_1\, d\alpha_2 \mathrm{e}^{-\left(\frac{m_e^2}{eE}\alpha_1 + \frac{M_d^2}{eE}\alpha_2 - \frac{M_m^2}{eE}\frac{\sin\alpha_1\ \sin\alpha_2}{\sin(\alpha_1+\alpha_2)}\right)}}{\alpha_1\ \sinh(\frac{g}{e}\alpha_2)\ \sin(\alpha_1+\alpha_2)\left(\frac{e}{\alpha_1} + g\cot(\frac{g}{e}\alpha_2) + g \right.}$$

$$\times\left[m_e M_d \cosh\left(\frac{g\alpha_2}{e}\right)\cos(\alpha_1-\alpha_2) + M_m^2\cosh\left(\frac{g\alpha_2}{e}\right)\frac{\sin\alpha_1\ \sin\alpha_2}{\sin^2(\alpha_1+\alpha_2)}\right].$$

Saddle-Point Integral 2

Custom integration via methods of the theory of complex variable functions **fails**, due to an essential non-analyticity of the integrand in α_1, α_2, like $e^{-1/x}$ in the vicinity of $x = 0$. One employs 2-dimensional saddle-point method for $\int d\alpha_1 d\alpha_2$. Minimizing $f(\alpha_1\,,\alpha_2) = \frac{m_e^2}{eE}\alpha_1 + \frac{M_d^2}{eE}\alpha_2 - \frac{M_m^2}{eE}\frac{\sin\alpha_1\,\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}$ one gets

$$\begin{pmatrix} \theta_1^{\pm(n)} \\ \theta_2^{\pm(m)} \end{pmatrix} = \pm \begin{pmatrix} \cos^{-1}\frac{M_m^2+m_e^2-M_d^2}{2m_e M_m} \\ \cos^{-1}\frac{M_m^2-m_e^2+M_d^2}{2M_d M_m} \end{pmatrix} + \begin{pmatrix} 2\pi n \\ 2\pi m \end{pmatrix} \equiv \begin{pmatrix} \pm\theta_1 + 2\pi n \\ \pm\theta_2 \pm 2\pi m \end{pmatrix}$$

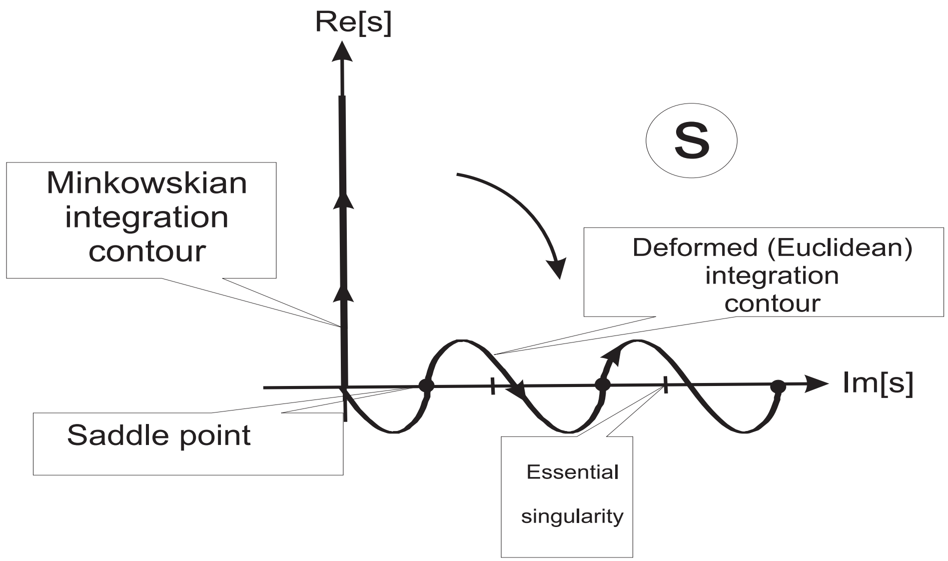
$n,\ m \in \mathbb{Z},\ \ \theta_i^{\pm(n)} > 0$, the corresponding determinant being

$$\det{}_{ij}\left(\frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j}\right) = -4\frac{\sin^2\theta_1\ \sin^2\theta_2}{\sin^4(\theta_1+\theta_2)}\left(\frac{M_m^2}{eE}\right)^2 = -4\frac{(m_e m_d)^2}{(eE)^2}.$$

Geometrically, the integer parameters m, n denote winding numbers of classical solutions.

addle-Point Integral: Contour Deformatio

Integration over α_1, α_2 contained a complicated contour rotation in $\mathbb{C}^2$. Below we show a simplified picture of how it should be done for one complex variable $s \in \mathbb{C}$



Here singularities do not lie on integration path; and saddle-points are passed in the (imaginary) direction prescribed by steepest descent condition. The deformation was performed in the domain of analyticity of the integrand, without traversing the singularities.

Sum Over Winding Numbers

Finally one obtains the mass correction as a sum over winding numbers m, n

$$\mathrm{Im}\,\delta M_m = -\frac{\lambda^2}{8\pi}\,\frac{eE}{M_m}\,\Bigg\{\sum_{n=0,\,m=0} \frac{\mathrm{e}^{-S^{+}_{n,m}}\ \cos^2(\frac{\theta_1-\theta_2}{2})}{\sin(\theta_1+\theta_2)\big(\frac{e}{\theta_1+2\pi n} + g\cot(\frac{g}{e}(\theta_2+2\pi m)) + g}$$

$$\times\frac{g}{(\theta_1+2\pi n)\tanh(\frac{g}{e}(\theta_2+2\pi m))} -$$

$$-\sum_{n=1,\,m=1} \frac{\mathrm{e}^{-S^{-}_{n,m}}\ \cos^2(\frac{\theta_1-\theta_2}{2})}{\sin(\theta_1+\theta_2)\big(\frac{e}{2\pi n-\theta_1} + g\cot(\frac{g}{e}(2\pi m-\theta_2)) + g\big)} \times$$

$$\times\frac{g}{(2\pi n-\theta_1)\tanh(\frac{g}{e}(2\pi m-\theta_2))}\Bigg\}$$, where

$$S^{+}_{n,m} = \frac{m_e^2}{eE}(\theta_1+2\pi n) + \frac{M_d^2}{eE}(\theta_2+2\pi m) - \frac{m_e M_d}{eE}\sin(\theta_1+\theta_2),$$

$$S^{-}_{n,m} = \frac{m_e^2}{eE}(2\pi n-\theta_1) + \frac{M_d^2}{eE}(2\pi m-\theta_2) + \frac{m_e M_d}{eE}\sin(\theta_1+\theta_2).$$ The leading term:

$$\mathrm{Im}\,\delta M_m = -\frac{\lambda^2}{4\sqrt{2}\pi}\,\frac{eE}{M_m}\,\mathrm{e}^{-S_0}\,\frac{\cos^2(\frac{\theta_1-\theta_2}{2})}{\sin(\theta_1+\theta_2)\big(\frac{e}{\theta_1} + g\cot(\frac{g}{e}\theta_2) + g\big)}\,\frac{g}{\theta_1\tanh(\frac{g}{e}\theta_2)},$$ with S_0

$$S_0 = \frac{m_e^2}{eE}\theta_1 + \frac{M_d^2}{eE}\theta_2 - \frac{m_e M_d}{eE}\sin(\theta_1+\theta_2).$$

Why semiclassical?

The leading exponential term in δM in the previous slide was obtained via approximating the Schwinger proper-time integrals by saddle-point approximation. What is the physical meaning of it?

In fact, when we got

$$S_0 = \frac{m_e^2}{eE}\theta_1 + \frac{M_d^2}{eE}\theta_2 - \frac{m_e M_d}{eE}\sin(\theta_1+\theta_2).$$

we have recovered the classical action

$$S_c = const_1 \cdot \text{Length} - const_2 \cdot \text{Area}$$

typical both for induced Schwinger processes in the first-quantized approach and for the 2D vacuum decay phenomena.

Terms proportional to $\theta_1, \theta_2 \sim$ *world-line length*.

Terms proportional to $\sin(\theta_1+\theta_2) \sim$ *area between world-lines*.

In general, our result amounts to an explicit verification of a general statement on *classical action* in ST and Green functions in FT

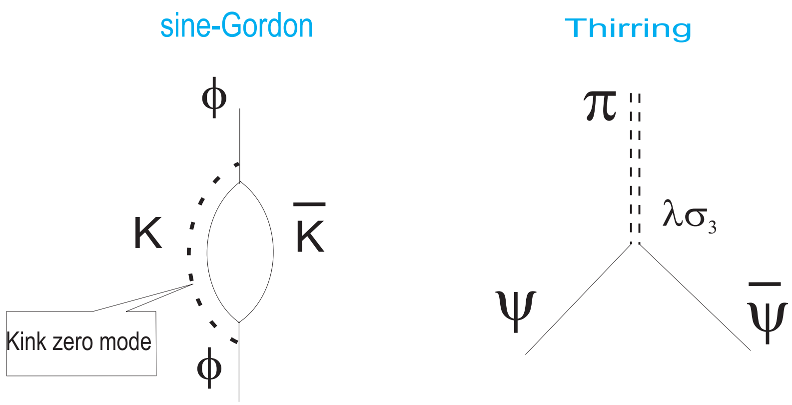
$$e^{-S_{\text{string theory}}}|_{classical} \sim \Sigma_{\text{field theory}}|_{semiclassical}$$

Thirring Model vs Sine-Gordon

For an induced Schwinger process in Thirring model there exists a calculation of the sub-leading fac
by Gorsky and Voloshin, based on vacuum decay in the dual theory (Sine-Gordon Model)

$$\Gamma = \frac{4g\mu}{\pi^3} e^{-S_0},$$

here g Thirring coupling constant, $g \gg 1$; μ mass of Thirring fermions, S_0 classical action.



K, $\bar{K}$ are kink-antikink pair in sine-Gordon model. LHS figure depicts a vacuum bubble with “legs” corresponding to in
ϕ particle, RHS – Schwinger decay of a meson π.

First bound state π of massive Thirring model is a pseudocsalar, because the fermionic current $j^\mu = \bar{\psi}\gamma^\mu\psi$ in Thirring model corresponds to a pseudovector quantity $\epsilon^{\mu\nu}\partial_\nu\phi$ in sine-Gordon model.

2D calculation

The suggested treatment of monopoles in 4D corresponds to the decay of bound state into a pair of a fermion and an antifermion of masses μ_1, μ_2 in 2D. It yields after resummation,

which is done exactly

$$\begin{aligned}\mathrm{Im}\,\delta m \quad = \quad & -\frac{\lambda^2}{4m}\frac{1}{\left(1-e^{-\frac{2\pi\mu_1^2}{eE}}\right)\left(1-e^{-\frac{2\pi\mu_2^2}{eE}}\right)\sin(\theta_1+\theta_2)}\times \\ & \times\left\{\mathrm{e}^{-S_0^+}\left[2\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{eE}{\mu_1\mu_2}\frac{1}{\sin(\theta_1+\theta_2)}\right]- \right.\\ & \left. -\,\mathrm{e}^{-S_0^-}\left[2\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)+\frac{eE}{\mu_1\mu_2}\frac{1}{\sin(\theta_1+\theta_2)}\right]\right\},\end{aligned}$$

and

$$\theta_1 = \cos^{-1}\frac{m^2+\mu_1^2-\mu_2^2}{2m\mu_1}$$

$$\theta_2 = \cos^{-1}\frac{m^2-\mu_1^2+\mu_2^2}{2m\mu_2}$$

$$S^+ = \frac{\mu_1^2}{eE}\theta_1 + \frac{\mu_2^2}{eE}\theta_2 - \frac{\mu_1\mu_2}{eE}\sin(\theta_1+\theta_2),$$

$$S^- = \frac{\mu_1^2}{eE}(2\pi-\theta_1) + \frac{\mu_2^2}{eE}(2\pi-\theta_2) + \frac{\mu_1\mu_2}{eE}\sin(\theta_1+\theta_2)$$

acuum Decay in SG=Schwinger in Thirri

In Thirring model, the above calculations for decay of bound state with mass m into two fermions with equal masses μ lead us to

$$\mathrm{Im}\,\delta m = -\frac{\lambda^2}{4m}\frac{\mathrm{e}^{-S_0}}{\sin 2\theta}\left(2 - \frac{eE}{\mu^2}\frac{1}{\sin 2\theta}\right), \text{ where } \quad \theta = \cos^{-1}\frac{m}{2\mu}$$

(resummation factor $\frac{1}{(1-e^{-\frac{2\pi\mu^2}{eE}})^2}$ omitted here). Let the external meson be the lightest bound state, then $m = \frac{\pi^2\mu}{2g} \ll \mu$. Comparison of Schwinger and vacuum decay yields

$$\lambda = \mu\sqrt{\frac{\pi}{g}}$$

This suggests a *perturbative interpretation of the non-perturbative result*: $1/\sqrt{g}$ **being small,** λ **effectively has a meaning of coupling constant** *in induced Schwinger process* **for the lightest Thirring meson.**

Finite Temperature: General Features

Several simple features are characteristic for theories at finite temperatures

- Green functions become periodic in Euclidean time, continuous momenta p_0 substituted for discrete Matsubara frequencies $\frac{2\pi n}{\beta}$
- An additional gauge-invariant quantity $e^{ie \oint A_\mu dx^\mu}$ (holonomy) characterizes the observables.
- The class of gauge transformations admitted by the theory is restricted to periodic functions in Euclidean time. However, *no* periodicity condition is imposed upon $A_\mu(x)$

Thus one can treat finite-temperature filed theory as a theory on a $\mathbb{R}^3 \times S^1$, with the restrictions above imposed.

Finite-Temperature Green Functions

We are going to consider again Thirring meson in 2 dimensions. Decay rate into a fermion-antifermion pair in an external field is calculated. This process has all characteristic properties of what happens to monopoles in 4D.

The Euclidean Green function for a charged particle in an external field $\vec{E} = (0, E)$ is expressed in terms of the following sum:

$$G(x,y) = \sum_{p=-\infty}^{\infty} \int \frac{eEds}{\sinh(eEs)} \times$$
$$\times\, e^{\frac{-i(x_0-y_0-p\beta)^2 eE\coth(eEs)}{4} - \frac{i(x_1-y_1)^2 eE\coth(eEs)}{4}} \times$$
$$\times e^{\frac{i}{2}eE(x_3-y_3)(x_0+y_0+p\beta)}$$

The imaginary part of meson mass is calculated in the same formalism as above.

hermal Corrections to Schwinger Process

We give an exact expression for the decay rate Γ It can be expressed in terms of a series in Matsubara frequencies:

$$\Gamma = \frac{\lambda^2 \beta^{-1} \epsilon^{-3/2}}{4m\sqrt{\pi}} \int \frac{d\alpha_1 d\alpha_2}{\sqrt{\sinh(\alpha_1+\alpha_2)\cosh(\alpha_1-\alpha_2)}} \times$$

$$\times \sum_{r,s\in\mathbb{Z}} \delta_{k+r+s} e^{\frac{4\pi^2 i}{eE}\frac{(r\tanh(\alpha_1)-s\tanh(\alpha_2))^2\sinh(2\alpha_1)\sinh(2\alpha_2)}{4\sinh(\alpha_1+\alpha_2)\cosh(\alpha_1-\alpha_2)}} \times$$

$$\times e^{i\left[r^2\tanh(\alpha_1)+s^2\tanh(\alpha_2)+\frac{\mu^2}{eE}(\alpha_1+\alpha_2)-\frac{m^2}{eE}\frac{1}{\coth(\alpha_1)+\coth(\alpha_2)}\right]}$$

where after doing summation one makes an analytic extension to continuous values of k and imposes $k=\frac{m\beta}{2\pi}$. The α_1, α_2 integrals are to be estimated by the saddle-point method.

“Duality”

Doing calculations with Matsubara sums, one makes extensive use of the well-known Poisson formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \tilde{f}(2\pi n)$$

where $\tilde{f}(k) = \int_{-\infty}^{\infty} f(t) e^{-ikt} dt$.

Therefore, either for $\beta \to 0$ or for $\beta \to \infty$ the leading term is the one with zero Matsubara frequency, and the sub-leading term (the first Matsubara frequency) is exponentially suppressed (like $\sim e^{-\frac{1}{\beta^2 eE}}$ or $\sim e^{-\beta^2 eE}$ respectively). More accurately, this “duality” manifests itself via the possibility to use the two equivalent series:

$$\Gamma \sim \frac{1}{\beta} \sum_s \int d\alpha_1 d\alpha_2 e^{i\left[\frac{\mu^2}{eE}(\alpha_1+\alpha_2) - \frac{m^2}{eE}\frac{1}{\coth(\alpha_1)+\coth(\alpha_2)} + \frac{4\pi^2 A}{eE\beta^2}(s-s_0)^2\right]} =$$

$$= \sum_s \int d\alpha_1 d\alpha_2 e^{i\left[\frac{\mu^2}{eE}(\alpha_1+\alpha_2) - \frac{m^2}{eE}\frac{1}{\cot h(\alpha_1)+\coth(\alpha_2)} - \frac{eE\beta^2 s^2}{4A} - 2\pi s s_0\right]}$$

dependent on the particular asymptotics. Here $A = \frac{\sinh(\alpha_1+\alpha_2)}{\cosh(\alpha_1-\alpha_2)}$, $s_0 = -k\frac{1}{\tanh(\alpha_2)(\coth(\alpha_1)+\coth(\alpha_2))}$, after the analytic continuation $k = \frac{m\beta}{2\pi}$.

Saddle Point vs. Matsubara Sum

The integrand series in the Schwinger proper-time integrals has two equivalent representations

$$(1) \sim e^{i\left[\frac{\mu^2}{eE}(\alpha_1+\alpha_2)-\frac{m^2}{eE}\frac{1}{\coth(\alpha_1)+\coth(\alpha_2)}+\frac{4\pi^2 A}{eE\beta^2}(s-s_0)^2\right]}$$

$$(2) \sim e^{i\left[\frac{\mu^2}{eE}(\alpha_1+\alpha_2)-\frac{m^2}{eE}\frac{1}{\cot h(\alpha_1)+\coth(\alpha_2)}-\frac{eE\beta^2 s^2}{4A}-2\pi s s_0\right]}$$

(Summation over s omitted above.)

"Duality" property summarized below:

	$\beta^2 eE \to 0$	$\beta^2 eE \to \infty$
L.O.	$\frac{\mu^2}{eE}(\alpha_1+\alpha_2) - \frac{m^2}{eE}\frac{1}{\coth(\alpha_1)+\coth(\alpha_2)} + \frac{4\pi^2 A}{eE\beta^2}(s_0)^2$	$\frac{\mu^2}{eE}(\alpha_1+\alpha_2) - \frac{m^2}{eE}\frac{1}{\coth(\alpha_1)+\mathrm{co}}$
N.L.O.	$\frac{4\pi^2 A}{eE\beta^2}$	$\frac{eE\beta^2}{4A} - 2\pi s_0$

Low-Temperature Limit

In the low-temperature limit for equal-mass fermions the zero-temperature result is reconstructed in the leading order,

$$\Gamma_{\mathrm{L.O.}} = \frac{\lambda^2 eE}{8\pi m}\frac{1}{\sin(2\overline{\alpha})} e^{-2\frac{\mu^2}{eE}\overline{\alpha}+\frac{m^2}{eE}\frac{1}{2\cot(\overline{\alpha})}}$$

where evaluating the saddle-point integrals is done at the same values $\alpha = i\overline{\alpha}$ as before

$$\cos\overline{\alpha} = \frac{m}{2\mu}.$$

High-Temperature Limit

Leading-order term prefactor is proportional to the temperature in the asypmtotic regime $\beta^2\epsilon \to 0$, $\mu \gg m$

$$\Gamma_{\text{L.O.}} \approx \frac{1}{\beta}\frac{\lambda^2}{\sqrt{eE}}\frac{1}{\sqrt{\sin(2\overline{\alpha})}}e^{-\frac{\mu^2}{eE}2\overline{\alpha}+\frac{m^2}{eE}\frac{1}{2\cot(\overline{\alpha})}}$$

Saddle-point value is different for this limit, namely

$$\overline{\alpha} \approx \frac{M}{m}$$

However, high-temperature regime requires more physical understanding: one should take care of distinguishing the competing purely quantum-mechanical tunnelling transitions, suppressed as $e^{-\frac{S_E}{\hbar}}$, and thermal (over-barrier) processes, suppressed as $e^{-\beta E}$. Therefore, we do not give next-to-leading order corrections here.

Main Results

- Monopole width up to the subleading semiclassical factor:

$$\mathrm{Im}\,\delta M_m \approx -\frac{\lambda^2}{4\sqrt{2}\pi}\frac{eE}{M_m}\frac{\mathrm{e}^{-S_0}\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)}{\sin(\theta_1+\theta_2)\left(\frac{e}{\theta_1}+g\cot\left(\frac{g}{e}\theta_2\right)+g\right)}\frac{g}{\theta_1\tanh\left(\frac{g}{e}\theta\right.}$$

- Effective "fermion–meson" vertex for Schwinger process in Thirring model:

$$\lambda = \mu\sqrt{\frac{\pi}{g}}.$$

- Finite-temperature corrections for meson decay width in an external field are calculated.

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