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Induced Gauge Theory on a Noncommutative Space

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Content

- Introduction

  nc scalar ϕ^4 theory, covariant coordinates, heat kernel, matrix basis

- Induced gauge action

  hep-th/0703169 with H. Grosse

  – Model in matrix basis
  – Technical details
  – Results and conclusions

- BRST quantisation

  work in progress with D. Blaschke, H. Grosse and M. Schweda

Intro

starting points:

canonically deformed Euclidean space: $[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}$

nc real scalar ϕ^4 theory ($\longrightarrow$ Grosse and Wulkenhaar)

$$S = \int d^Dx \left(\frac{1}{2}\phi \star [\tilde{x}_\nu,\, [\tilde{x}^\nu,\, \phi]_\star]_\star + \frac{\Omega^2}{2}\phi \star \{\tilde{x}^\nu, \{\tilde{x}_\nu, \phi\}_\star\}_\star + \frac{\mu^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right)(x) \,,$$

where $\tilde{x}_\nu = \theta^{-1}_{\nu\alpha} x^\alpha$ and $i\partial_\mu f = [\tilde{x}_\mu, f]_\star$

Intro

Remarks:

- discrete spectrum of the Hamiltonian
- no UV/IR mixing due to oscillator term
- theory renormalisable

Intro

covariant coordinates

let $\mathcal{G}$ be a unitary gauge group under which the scalar field ϕ transforms covariantly like

$$\phi \mapsto u^* \star \phi \star u, \;\; u \in \mathcal{G}.$$

The product $x^\mu \star \phi$ will not transform covariantly,

$$x^\mu \star \phi \nrightarrow u^* \star x^\mu \star \phi \star u \; .$$

$\Rightarrow$ covariant coordinates

$$\tilde{X}_\nu = \tilde{x}_\nu + A_\nu,$$

where

$$\begin{aligned} A_\mu &\mapsto \mathrm{i} u^* \star \partial_\mu u + u^* \star A_\mu \star u \, , \\ \tilde{X}_\mu &\mapsto u^* \star \tilde{X}_\mu \star u \end{aligned}$$

Intro

gauge invariant action

$$S = \int d^Dx \left(\frac{1}{2}\phi \star [\tilde{X}_\nu, [\tilde{X}^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2}\phi \star \{\tilde{X}^\nu, \{\tilde{X}_\nu, \phi\}_\star\}_\star + \frac{\mu^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right)(x) ,$$

where:

$$\begin{aligned} \phi &\mapsto u^* \star \phi \star u \,, \\ A_\mu &\mapsto \mathrm{i}u^* \star \partial_\mu u + u^* \star A_\mu \star u \,, \\ \tilde{X}_\mu &\mapsto u^* \star \tilde{X}_\mu \star u \,. \end{aligned}$$

Intro

heat kernel expansion

$$\Gamma^{\epsilon}_{1l}[\phi] = -\frac{1}{2}\int_{\epsilon}^{\infty}\frac{dt}{t}\,\mathrm{Tr}\left(e^{-tH} - e^{-tH^0}\right) \ .$$

where we use the effective potential

$$\frac{\delta^2 S}{\delta\phi^2} \equiv H = \frac{2}{\theta}H^0 + V$$

The method is not manifestly gauge invariant, contributions from different orders need to add up to a gauge invariant result.

Intro

We employ the Duhamel expansion which is an iteration of the identity

$$\begin{aligned} e^{-tH} - e^{-tH^0} &= \int_0^t d\sigma \, \frac{d}{d\sigma}\left(e^{-\sigma H} e^{-(t-\sigma)H^0}\right) \\ &= -\int_0^t d\sigma \; e^{-\sigma H} \, \frac{\theta}{2} V \, e^{-(t-\sigma)H^0} \; , \end{aligned}$$

giving

$$\begin{aligned} e^{-tH} &= e^{-tH^0} - \frac{\theta}{2}\int_0^t dt_1 e^{-t_1 H^0} V e^{-(t-t_1)H^0} \\ &\quad + \left(\frac{\theta}{2}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-t_2 H^0} V e^{-(t_1-t_2)H^0} V e^{-(t-t_1)H^0} + \ldots \end{aligned}$$

Intro

Therefore, we have for the 1-loop effective action is given by

$$
\begin{aligned}
\Gamma_{1l}^{\epsilon} \quad &= \quad -\frac{1}{2}\int_{\epsilon}^{\infty}\frac{dt}{t}\mathrm{Tr}\left(e^{-tH}-e^{-tH^{0}}\right) \\
&= \quad \frac{\theta}{4}\int_{\epsilon}^{\infty} dt\ \mathrm{Tr}\ V e^{-tH^{0}} - \frac{\theta^{2}}{8}\int_{\epsilon}^{\infty}\frac{dt}{t}\int_{0}^{t} dt'\, t'\ \mathrm{Tr}\ V e^{-t'H^{0}} V e^{-(t-t')H^{0}} \\
& \qquad +\frac{\theta^{3}}{16}\int_{\epsilon}^{\infty}\frac{dt}{t}\int_{0}^{t} dt'\int_{0}^{t'} dt''\, t''\ \mathrm{Tr}\ V e^{-t''H^{0}} V e^{-(t'-t'')H^{0}} V e^{-(t-t')H^{0}} \\
& -\frac{\theta^{4}}{32}\int_{\epsilon}^{\infty}\frac{dt}{t}\int_{0}^{t} dt'\int_{0}^{t'} dt''\int_{0}^{t''} dt'''\, t'''\ \mathrm{Tr}\ V e^{-t'''H^{0}} V e^{-(t''-t''')H^{0}} V e^{-(t'-t'')H^{0}} V e^{-(t-t')H^{0}} \\
&+ \quad +\mathcal{O}(\theta^{5})
\end{aligned}
$$

Intro

The action contains an oscillator term

$$\frac{\Omega^2}{2}\phi \star \{\tilde{X}^\nu, \{\tilde{X}_\nu, \phi\}_\star\}_\star \,.$$

This term is crucial, it alters the free theory.

Therefore, we expand around the free action $-\Delta + \Omega^2 \tilde{x}^2$ rather than $-\Delta$.
Seeley-de Witt coefficients cannot be used!!

Intro

The calculations are performed in the matrix basis, where the star product is just a matrix product, e.g. for $D = 4$:

$$A^{\nu}(x) = \sum_{p,q \in \mathbb{N}^2} A^{\nu}_{pq} f_{pq}(x) \;, \phi(x) = \sum_{p,q \in \mathbb{N}^2} \phi_{pq} f_{pq}(x)$$

and

$$f_{pq} \star f_{mn} = \delta_{qm} f_{pn},$$
$$f_0 \star f_0 = f_0$$

Intro

matrix basis, important relations

$$
\begin{aligned}
f_0(x) &= 4\, e^{-\frac{1}{\theta}\sum_i x_i^2}, \\
a^{(1)} &= \frac{1}{\sqrt{2}}(x^1 + ix^2), a^{(2)} = \frac{1}{\sqrt{2}}(x^3 + ix^4), \\
f_{\substack{m^1 n^1 \\ m^2 n^2}} &= \alpha(n, m, \theta)\, \bar{a}^{(2)\star m^2} \star \bar{a}^{(1)\star m^1} \star f_0 \star a^{(1)\star n^1} \star a^{(2)\star n^2}, \\
\bar{a}^{(1)} \star f_{\substack{m^1 n^1 \\ m^2 n^2}} &= \sqrt{\theta(m^1+1)} f_{\substack{m^1+1\, n^1 \\ m^2\ \ n^2}}, \\
a^{(1)} \star f_{\substack{m^1 n^1 \\ m^2 n^2}} &= \sqrt{\theta m^1} f_{\substack{m^1-1\, n^1 \\ m^2\ \ n^2}}
\end{aligned}
$$

The model

The part independent of the gauge field in the matrix basis is given by

$$\begin{aligned}\frac{H^0_{mn;kl}}{1+\Omega^2} &= \Big(\frac{\tilde{\mu}^2}{2}+(n^1+m^1+1)+(n^2+m^2+1)\Big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad -\rho\Big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1}+\sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\Big)\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad -\rho\Big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2}+\sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\Big)\delta_{n^1k^1}\delta_{m^1l^1}\end{aligned}$$

All the operators depend beside on θ only on the following three parameters:

$$\rho=\frac{1-\Omega^2}{1+\Omega^2},\ \tilde{\epsilon}=\epsilon(1+\Omega^2),\ \tilde{\mu}^2=\frac{\mu^2\theta}{1+\Omega^2}$$

$$
\begin{aligned}
V_{kl;mn}(1+\Omega^2) \quad = \quad & \Big(\frac{\lambda}{3!(1+\Omega^2)}\phi\star\phi + \big(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2\big)\Big)_{lm}\delta_{nk} \\
& +\Big(\frac{\lambda}{3!(1+\Omega^2)}\phi\star\phi + \big(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2\big)\Big)_{nk}\delta_{lm} \\
& +\Big(\frac{\lambda}{3!(1+\Omega^2)}\phi_{lm}\phi_{nk} - 2\rho A_{\nu,lm}A^\nu_{nk}\Big) \\
& +\rho\mathrm{i}\sqrt{\frac{2}{\theta}}\Big(\sqrt{n^1}A^{(1+)}_{\substack{l^1m^1\\ l^2m^2}}\delta_{\substack{n^1-1k^1\\ n^2\ k^2}} - \sqrt{n^1+1}A^{(1-)}_{\substack{l^1m^1\\ l^2m^2}}\delta_{\substack{n^1+1k^1\\ n^2\ k^2}} \\
& \qquad + \sqrt{n^2}A^{(2+)}_{\substack{l^1m^1\\ l^2m^2}}\delta_{\substack{n^1\ k^1\\ n^2-1k^2}} - \sqrt{n^2+1}A^{(2-)}_{\substack{l^1m^1\\ l^2m^2}}\delta_{\substack{n^1\ k^1\\ n^2+1k^2}}\Big) \\
& -\rho\mathrm{i}\sqrt{\frac{2}{\theta}}\Big(- \sqrt{m^1+1}A^{(1+)}_{\substack{n^1k^1\\ n^2k^2}}\delta_{\substack{l^1m^1+1\\ l^2\ m^2}} + \sqrt{m^1}A^{(1-)}_{\substack{n^1k^1\\ n^2k^2}}\delta_{\substack{l^1m^1-1\\ l^2\ m^2}} \\
& \qquad - \sqrt{m^2+1}A^{(2+)}_{\substack{n^1k^1\\ n^2k^2}}\delta_{\substack{l^1\ m^1\\ l^2m^2+1}} + \sqrt{m^2}A^{(2-)}_{\substack{n^1k^1\\ n^2k^2}}\delta_{\substack{l^1\ m^1\\ l^2m^2-1}}\Big)
\end{aligned}
$$

with

$$
A^{(1\pm)} = A^1 \pm iA^2,\ A^{(2\pm)} = A^3 \pm iA^4
$$

The model

$$\left(e^{-tH^0}\right)_{mn;kl} = e^{-t(\mu^2\theta/2+4\Omega)}\delta_{m+k,n+l}\prod_{i=1}^{2} K_{m^i n^i;k^i l^i}(t) \; ,$$

$$K_{m,m+\alpha;l+\alpha,l}(t) = \sum_{u=0}^{\min(m,l)} \sqrt{\binom{m}{u}\binom{l}{u}\binom{\alpha+m}{m-u}\binom{\alpha+l}{l-u}} \times e^{2\Omega t}\left(\frac{1-\Omega^2}{2\Omega}\sinh(2\Omega t)\right)^{m+l-2u} X_\Omega(t)^{\alpha+m+l+1} \; ,$$

where

$$X_\Omega(t) = \frac{4\Omega}{(1+\Omega)^2 e^{2\Omega t} - (1-\Omega)^2 e^{-2\Omega t}} \; .$$

Technical details

Again the Duhamel expansion of the effective action

$$
\begin{aligned}
\Gamma^{\epsilon}_{1l} &= -\frac{1}{2}\int_{\epsilon}^{\infty}\frac{dt}{t}\mathrm{Tr}\left(e^{-tH}-e^{-tH^0}\right) \\
&= \frac{\theta}{4}\int_{\epsilon}^{\infty}dt\ \mathrm{Tr}\ Ve^{-tH^0}-\frac{\theta^2}{8}\int_{\epsilon}^{\infty}\frac{dt}{t}\int_0^t dt'\,t'\ \mathrm{Tr}\ Ve^{-t'H^0}Ve^{-(t-t')H^0} \\
&+ \frac{\theta^3}{16}\int_{\epsilon}^{\infty}\frac{dt}{t}\int_0^t dt'\int_0^{t'}dt''\,t''\ \mathrm{Tr}\ Ve^{-t''H^0}Ve^{-(t'-t'')H^0}Ve^{-(t-t')H^0} \\
&- \frac{\theta^4}{32}\int_{\epsilon}^{\infty}\frac{dt}{t}\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}dt'''\,t'''\ \mathrm{Tr}\ Ve^{-t'''H^0}Ve^{-(t''-t''')H^0}Ve^{-(t'-t'')H^0}Ve^{-(t-t')H^0} \\
&+ +\mathcal{O}(\theta^5)
\end{aligned}
$$

insert all the above expressions!

Technical details

We concentrate on the divergent terms involving only the gauge field, therefore $\lambda = 0$

Method to extract divergent contributions:

- Expand the integrands for small auxiliary parameters
- Expand infinite sums over indices occuring in the heat kernel but not in the gauge fields; divergences stem from these infinite sums
- Integrate over the auxiliary parameters
- Convert the results to x-space using

$$\sum_m T_{mm} = \frac{1}{(2\pi\theta)^2} \int d^4x T(x)$$

Technical details

Example from 2nd order:

$$\frac{\theta}{4}\int_{\epsilon}^{\infty}\frac{dt}{t}e^{-2t\sigma^2}\int_0^t dt' t' \, (1-\Omega^2)^2$$
$$\times \sum (k^1+1) A^{(1+)}_{\substack{c^1 \\ c^2}\substack{m^1+1 \\ m^2}} A^{(1+)}_{\substack{m^1 \\ m^2}\substack{c^1+1 \\ c^2}} K_{\substack{k^1+1 \\ k^2}, \substack{m^1+1 \\ m^2}; \substack{m^1 \\ m^2}, \substack{k^1 \\ k^2}}(t') K_{\substack{k^1+1 \\ k^2}, \substack{l^1+1 \\ l^2}; \substack{l^1 \\ l^2}, \substack{k^1 \\ k^2}}(t-t')$$
$$= -\frac{\theta}{24}\left(\frac{1-\Omega^2}{1+\Omega^2}\right)^4 \ln\epsilon\sqrt{m^1+1}\sqrt{c^1+1}\, A^{(1+)}_{\substack{c^1 \\ c^2}\substack{m^1+1 \\ m^2}} A^{(1+)}_{\substack{m^1 \\ m^2}\substack{c^1+1 \\ c^2}}$$

Technical details

using the relations

$$\sum_{n=0}^{\infty} K_{nm;mn}(t') K_{n,c;c,n}(t-t') \quad \propto \quad \frac{1}{t} + \mathcal{O}(t^0, t'^0)$$

$$\sum_{n=0}^{\infty} \sqrt{n+1} K_{n+1,m+1;mn}(t') K_{n+1,c+1;c,n}(t-t') \quad \propto \quad \frac{t'(t-t')}{t^3} + \mathcal{O}(t'^0, t^0)$$

Technical details

- first and second order: quadratic and logarithmic divergences
- third and fourth order: logarithmic divergences
- other orders are finite

Results

$D = 2$, $\Omega = 1$

$$\Gamma^{\epsilon}_{1l} = \frac{-1}{4\pi} \int d^2x (\tilde{X}_\mu \star \tilde{X}^\mu - \tilde{x}^2) \ln \epsilon$$

$D = 2$, $\Omega \neq 1$

$$\Gamma^{\epsilon}_{1l} = \frac{-2\Omega^2}{2\pi(1+\Omega^2)^2} \int d^2x \, (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \ln \epsilon$$

Results

$D = 4$, $\Omega = 1$

$$\Gamma^{\epsilon}_{1l} = \frac{1}{16\pi^2}\int d^4x \left(\frac{1}{\epsilon\theta}(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) + \left(\frac{\mu^2}{2}(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) + \frac{1}{2}\Big((\tilde{X}_\mu \star \tilde{X}^\mu) \star (\tilde{X}_\nu \star \tilde{X}^\nu) - (\tilde{x}^2)^2\Big)\right) \ln\epsilon\right)$$

Results

$D = 4$, $\Omega \neq 1$

$$\Gamma^{\epsilon}_{1l} = \frac{1}{192\pi^2}\int d^4x \left\{ \frac{24}{\tilde{\epsilon}\,\theta}(1-\rho^2)(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) + \ln\epsilon\left(\frac{12}{\theta}(1-\rho^2)(\tilde{\mu}^2-\rho^2)(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) + 6(1-\rho^2)^2\left((\tilde{X}_\mu \star \tilde{X}^\mu)^{\star 2} - (\tilde{x}^2)^2\right) - \rho^4 F_{\mu\nu}F^{\mu\nu}\right)\right\},$$

where $F_{\mu\nu} = -i[\tilde{x}_\mu, A_\nu]_\star + i[\tilde{x}_\nu, A_\mu]_\star - i[A_\mu, A_\nu]_\star$

Discussion

- $\Omega \to 0$ ($\rho \to 1$): usual NCYM
- $\Omega \to 1$ ($\rho \to 0$): obtain interesting matrix models
- quantisation needs to be studied; problem with ocurring tadpole
- $\theta \to \infty$ limit possible
- sign of quadratic term depends on $\tilde{\mu}^2 \lesseqgtr \rho^2$ $\to$ phase structure
- result coincides with the one obtained by de Goursac, Wallet and Wulkenhaar

BRST

similar action proposed in 0705.4205[hep-th]

$$S = \int d^4x \Big(\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{\Omega^2}{8} \tilde{x}_\mu \star \{A_\nu \stackrel{\star}{,} \{A_\nu \stackrel{\star}{,} \tilde{x}_\mu\}\} \Big)$$

$$\downarrow$$

$$S = \int d^4x \Big(\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + s(\bar{c} \star \partial_\mu A_\mu) - \frac{1}{2} B^2 + \frac{\Omega^2}{8} s(\tilde{c}_\mu \star \mathcal{C}_\mu) \Big)$$

where $\mathcal{C}_\mu = \{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} A_\nu\} + [\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \stackrel{\star}{,} c] + [\bar{c} \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} c\}]$ and BRST transformations

$$\begin{aligned} sA_\mu &= D_\mu c, \quad s\bar{c} = B, \qquad sc = igc \star c, \\ sB &= 0, \qquad s\tilde{c}_\mu = \tilde{x}_\mu \end{aligned}$$

BRST

- Tadpole contributions do not disappear, but are finite; diverge for $\Omega \to 0$
- Classical vacuum?
- 1-loop calculations...
- Is this model related to the induced model? At 1-loop additional terms may appear and the induced action may be recovered