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Superalgebras of Dirac operators on manifolds admitting Killing-Yano tensors

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Plan of the talk

1. Killing tensors
2. Fermions in a pseudo-classical approach
3. Dirac equation on curved spaces
4. Kac-Moody algebras
5. Anomalies
6. Summary

- I. I. Cotăescu, M. V., *Fortschr. Phys.* **54** (2006) 1142.
- I. I. Cotăescu, M. V., *Symmetries and supersymmetries of the Dirac operators in curved spacetimes* in *Frontiers in General Relativity and Quantum Cosmology Research*, pp. 109-166, (Nova Science N.Y. 2007).
- I. I. Cotăescu, M. V., *J. Phys.A: Math. Theor*, **42** (2007) (in press); *arXiv: 0705.0866 [hep-th]*.

Killing tensors

1. Symmetric Stäckel-Killing (S-K) tensors

$$K_{(\mu\cdots\nu;\lambda)} = 0.$$

2. Antisymmetric Killing-Yano (K-Y) tensors

$$f_{\mu_1\ldots\mu_{r-1}(\mu_r;\lambda)} = 0.$$

In special cases Stäckel-Killing tensors can be expressed as symmetrized product of Killing-Yano tensors.

Pseudo-classical approach

F. A. Berezin, M. S. Marinov (1977)
G.W.Gibbons, R.H.Rietdijk, J.W.van Holten (1993).
Action:

$$S = \int_a^b d\tau \left(\frac{1}{2} g_{\mu\nu}(x)\, \dot{x}^\mu\, \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x)\, \psi^\mu\, \frac{D\psi^\nu}{D\tau} \right).$$

Covariant derivative of ψ^μ

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma^\mu_{\lambda\nu} \psi^\nu \,.$$

World-line Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu$$

covariant momentum

$$\Pi_\mu = g_{\mu\nu} \dot{x}^\nu$$

Constant of motion $\mathcal{J}(x, \Pi, \psi)$, the bracket with H vanishes

$$\{H, \mathcal{J}\} = 0 \,.$$

Expand $\mathcal{J}(x, \Pi, \psi)$ in a power series in the covariant momentum

$$\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}^{(n)\mu_1 \dots \mu_n}(x, \psi) \Pi_{\mu_1} \dots \Pi_{\mu_n}$$

Generalized Killing equations:

$$\mathcal{J}^{(n)}_{(\mu_1 \dots \mu_n ; \mu_{n+1})} + \frac{\partial \mathcal{J}^{(n)}_{(\mu_1 \dots \mu_n}}{\partial \psi^\sigma} \Gamma^\sigma_{\mu_{n+1})\lambda} \psi^\lambda = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\nu(\mu_{n+1}} \mathcal{J}^{(n+1)\nu}{}_{\mu_1 \dots \mu_n)}$$

For a Killing vector R_μ ($R_{(\mu;\nu)} = 0$) there is a conserved quantity in the spinning case:

$$\mathcal{J} = \frac{i}{2} R_{[\mu;\nu]} \psi^\mu \psi^\nu + R_\mu \dot{x}^\mu$$

Assume that a S-K tensor can be written as a symmetrized product of two K-Y tensors

$$K^{\mu\nu}_{ij} = \frac{1}{2} (f^\mu_{i\ \lambda} f^{\nu\lambda}_j + f^\nu_{i\ \lambda} f^{\mu\lambda}_j).$$

The conserved quantity for the spinning space is

$$\mathcal{J}_{ij} = \frac{1}{2!} K_{ij}^{\mu\nu} \dot{x}_\mu \dot{x}_\nu + \mathcal{J}_{ij}^{(1)\mu} \dot{x}_\mu + \mathcal{J}_{ij}^{(0)}$$

where

$$\mathcal{J}_{ij}^{(0)} = -\frac{1}{4} \psi^\lambda \psi^\sigma \psi^\rho \psi^\tau (R_{\mu\nu\lambda\sigma} f_i^{\mu}{}_{\rho} f_j^{\nu}{}_{\tau} + \frac{1}{2} c_{i\lambda\sigma}{}^{\pi} c_{j\rho\tau\pi}),$$

$$\mathcal{J}_{ij}^{(1)\mu} = \frac{i}{2} \psi^\lambda \psi^\sigma (f_i^{\nu}{}_{\sigma} D_\nu f_j^{\mu}{}_{\lambda} + f_j^{\nu}{}_{\sigma} D_\nu f_i^{\mu}{}_{\lambda} + \frac{1}{2} f_i^{\mu\rho} c_{j\lambda\sigma\rho} + \frac{1}{2} f_j^{\mu\rho} c_{i\lambda\sigma\rho})$$

with

$$c_{i\mu\nu\lambda} = -2 f_{i[\nu\lambda;\mu]}$$

Conserved supercharge

$$\begin{aligned} Q_f &= f_{\mu_1 \ldots \mu_r} \Pi^{\mu_1} \psi^{\mu_2} \ldots \psi^{\mu_r} \\ &\quad + \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \ldots \mu_r ; \mu_{r+1}]} \cdot \psi^{\mu_1} \ldots \psi^{\mu_{r+1}}. \end{aligned}$$

This quantity is a superinvariant (supercharge $Q_0 = \Pi_\mu \psi^\mu$)

$$\{Q_f, Q_0\} = 0$$

Dirac equation on a curved background

B. Carter (1977).
B. Carter, R. G. McLenaghan (1979).
R. G. McLenaghan, Ph. Spindel (1979).

Dirac operator on a curved background $(\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I)$

$$D_s = \gamma^\mu \hat{\nabla}_\mu .$$

Canonical covariant derivative for spinors

$$\hat{\nabla}_\mu \gamma^\mu = 0,$$

$$\hat{\nabla}_{[\rho} \hat{\nabla}_{\mu]} = \frac{1}{4} R_{\alpha\beta\rho\mu} \gamma^\alpha \gamma^\beta$$

For any isometry with Killing vector R_μ there is an operator

$$X_k = -i(R^\mu \hat{\nabla}_\mu - \frac{1}{4}\gamma^\mu \gamma^\nu R_{\mu;\nu})$$

which commutes with the *standard* Dirac operator.

A K-Y tensor produces a *non-standard* Dirac operator

$$D_f = -i\gamma^\mu (f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6}\gamma^\nu \gamma^\rho f_{\mu\nu;\rho})$$

which anticommutes with the standard Dirac operator D_s.

Covariantly constant K-Y tensors

Definition

The non-singular real or complex-valued K-Y tensor f of rank 2 defined on M_n which satisfies

$$f^{\mu\cdot}_{\cdot\alpha} f_{\mu\beta} = g_{\alpha\beta} \, ,$$

is called an unit root of the metric tensor of M_n

Any unit root of the metric tensor is a covariantly constant K-Y tensor.

Theorem

The Dirac-type operator D_f produced by the K-Y tensor f satisfies the condition

$$(D_f)^2 = D^2$$

if and only if f is an unit root.

Non covariantly constant K-Y tensors

Hidden symmetries !

Euclidean Taub-NUT space

S. W. Hawking (1977).
D. Gross, M. J. Perry (1983).
R. D. Sorkin (1983).
M. F. Atiyah, N. Hitchin (1985).
G. W. Gibbons, N. S. Manton (1986).
G. W. Gibbons, P. J. Ruback (1988).
L. Gy. Feher, P. A. Horvathy (1987).
B. Cordani, L. Gy. Feher, P. A. Horvathy (1988).

Metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(r)(d\vec{x})^2 + \frac{g(r)}{\mu^2}(dx^4 + A_i^{em} dx^i)^2$$

where μ is a parameter, $\overrightarrow{A}^{em}$ is the gauge field of a monopole

$$\text{div}\vec{A}^{em} = 0\,, \quad \vec{B}^{em} = \text{rot}\,\vec{A}^{em} = \mu\frac{\vec{x}}{r^3}.$$

$$f(r) = g^{-1}(r) = V^{-1}(r) = \frac{\mu + r}{r}$$

In spherical coordinates

$$ds^2 = f(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2) + g(r)(d\chi + \cos\theta \, d\varphi)^2$$

where the angle variables (θ, φ, χ) parametrize the sphere S^3 with

$$0 \leq \theta < \pi, 0 \leq \varphi < 2\pi, 0 \leq \chi < 4\pi.$$

In the Cartesian charts the tetrads are:

$$\hat{e}^i_j = \frac{1}{\sqrt{V}} \delta_{ij} \, , \quad \hat{e}^4_i = \sqrt{V} A^{em}_i \, , \quad \hat{e}^4_4 = \sqrt{V} \, .$$

Four Killing vectors

$$D_A = R^\mu_A \,\partial_\mu, \qquad A = 1, 2, 3, 4\,.$$

Conservation of angular momentum and "relative electric charge" ($\vec{p} = V^{-1}\dot{\vec{r}}$ is the mechanical momentum):

$$\vec{j} = \vec{r} \times \vec{p} + q\,\frac{\vec{r}}{r}\,,$$
$$q = g(r)(\dot{\theta} + \cos\theta\dot{\varphi})\,.$$

Four K-Y tensors of valence 2.

- Three are covariantly constant

$$f^{(i)} = f^{(i)}_{\hat{\alpha}\hat{\beta}} \hat{e}^{\hat{\alpha}} \wedge \hat{e}^{\hat{\beta}} = 2\hat{e}^i \wedge \hat{e}^4 - \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k \,,$$

- The fourth K-Y tensor is

$$f^Y = f^Y_{\hat{\alpha}\hat{\beta}} \hat{e}^{\hat{\alpha}} \wedge \hat{e}^{\hat{\beta}} = \frac{x^i}{r} f^{(i)} + \frac{2x^i}{\mu V} \varepsilon_{ijk} \, \hat{e}^j \wedge \, \hat{e}^k \,,$$

  having a non-vanishing covariant derivative

The presence of f^Y is related to the existence of the hidden symmetries.

Runge-Lenz vector

$$\vec{K} = \frac{1}{2}\,\vec{K}_{\mu\nu}\,\dot{x}^{\mu}\dot{x}^{\nu} = \vec{p} \times \vec{j} + \left(\frac{q^2}{\mu} - \mu E\right)\frac{\vec{r}}{r}$$

where

$$E = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

is the energy.

The components $K_{i\mu\nu}$ are **Stäckel-Killing tensors**

$$K_{i\mu\nu} - \frac{1}{2\mu}(R_{4\mu}R_{i\nu} + R_{4\nu}R_{i\mu}) = \frac{\mu}{4}\left(f_{Y\,\mu\lambda}f_{i}{}^{\lambda}{}_{\nu} + f_{Y\,\nu\lambda}f_{i}{}^{\lambda}{}_{\mu}\right).$$

Scalar quantum mechanics

Momenta: $P_4 = -i\partial_4$ and $P_i = -i(\partial_i - A_i^{em}\partial_4)$.
Angular momentum

$$\vec{L} \, = \, \vec{x} \times \vec{P} - \mu \frac{\vec{x}}{r} P_4 \, .$$

These operators obey

$$[P_i, P_j] = i\varepsilon_{ijk} B_k^{em} P_4$$
$$[P_i, P_4] = 0$$
$$[L_i, \, P_j] = i\varepsilon_{ijk} P_k$$

Klein-Gordon operator

$$\Delta = V \vec{P}^{\,2} + \frac{1}{V} {P_4}^2 \, ,$$

Runge-Lenz vector

$$\vec{K} = -\frac{1}{2}\nabla_\mu \vec{k}^{\mu\nu}\nabla_\nu = \frac{1}{2}\left(\vec{P}\times\vec{L} - \vec{L}\times\vec{P}\right) - \mu\frac{\vec{x}}{r}\left(\frac{1}{2}\Delta - P_4^2\right),$$

Commutation relations

$$\begin{aligned}[L_i,\, L_j] &= i\varepsilon_{ijk}\, L_k\,, \\ [L_i,\, K_j] &= i\varepsilon_{ijk}\, K_k\,, \\ [K_i,\, K_j] &= i\varepsilon_{ijk}\, L_k B^2\,,\end{aligned}$$

where

$$B^2 = {P_4}^2 - \Delta.$$

Casimir operators

$$C_1 = \vec{L}^2 B^2 + \vec{K}^2 = \mu^2 P_4^2 B^2 + \frac{\mu^2}{4}\left(B^2 + P_4{}^2\right)^2 - B^2$$

$$C_2 = \vec{L} \cdot \vec{K} = -\frac{\mu^2}{2} P_4 (B^2 + P_4^2)\,.$$

New Casimir operators

$$C^{\pm} = C_1 \pm 2BC_2 + B^2 = \frac{\mu^2}{4}(P_4 \mp B)^4 = B^2 (N \pm \mu P_4)^2\,.$$

where N is the operator whose eigenvalues are just the values of the principal quantum number of the discrete energy spectra.

Dirac theory on Taub-NUT

I. I. Cotăescu, M. Visinescu (2000, 2001, 2004).

Dirac matrices $\gamma^{\hat{\alpha}}$ satisfy

$$\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta_{\hat{\alpha}\hat{\beta}}$$

.

Representation (σ_i are Pauli matrices)

$$\gamma^i = -i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}.$$

In addition

$$\gamma^0 = i\gamma^1\gamma^2\gamma^3\gamma^4 = i\,\mathrm{diag}(\mathbf{1}_2, -\mathbf{1}_2)$$

.

Standard Dirac operator without explicit mass term is

$$D = \gamma^\alpha \nabla_\alpha$$

Massless Hamiltonian operator

$$H = -i\gamma^0 D = \begin{pmatrix} 0 & \boldsymbol{\alpha}^* \\ \boldsymbol{\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & V\pi^* \frac{1}{\sqrt{V}} \\ \sqrt{V}\pi & 0 \end{pmatrix},$$

where

$$\pi = \sigma_P - iV^{-1}P_4 \ , \pi^* = \sigma_P + iV^{-1}P_4 \ , \sigma_P = \vec{\sigma} \cdot \vec{P} \, .$$

Conserved operators

From an operator $\hat{X}$ which commutes with $\boldsymbol{\Delta} = \boldsymbol{\alpha}^* \boldsymbol{\alpha} = V \pi^* \pi$.

$$[\hat{X}, \boldsymbol{\Delta}] = 0$$

diagonal Dirac operator

$$\mathcal{D}(\hat{X}) = \begin{pmatrix} \hat{X} & 0 \\ 0 & \boldsymbol{\alpha} \hat{X} \boldsymbol{\Delta}^{-1} \boldsymbol{\alpha}^* \end{pmatrix},$$

is also conserved.

Another type of conserved Dirac operator

$$\mathcal{Q}(\hat{X}) = \left\{ H \, , \begin{pmatrix} \hat{X} & 0 \\ 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & \hat{X} \boldsymbol{\alpha}^* \\ \boldsymbol{\alpha} \hat{X} & 0 \end{pmatrix},$$

Total angular momentum

$$\mathcal{J}_i = L_i + S_i\,, \quad S_i = \tfrac{1}{2}\varepsilon_{ijk}S^{jk} = \tfrac{1}{2}\mathrm{diag}(\sigma_i, \sigma_i)\,,$$

Equivalent

$$\mathcal{J}_i = \mathcal{D}(L_i) + \tfrac{1}{2}\mathcal{D}(\sigma_i)\,,$$

The triplet of Killing-Yano operators $\mathbf{f_i}$ gives rise to the spin-like operators

$$\Sigma^{(i)} = \frac{i}{4} f^{(i)}_{\hat\alpha\hat\beta}\gamma^{\hat\alpha}\gamma^{\hat\beta} = \left(\begin{array}{cc} \sigma_i & 0 \\ 0 & 0 \end{array}\right),$$

and produce the Dirac-type operators

$$D^{(i)} = -f^{(i)}_{\mu,\nu}\gamma^\nu\nabla^\mu = i[D,\,\Sigma^{(i)}] = -i\left(\begin{array}{cc} 0 & \sigma_i\boldsymbol{\alpha}^* \\ \boldsymbol{\alpha}\sigma_i & 0 \end{array}\right) = -i\mathcal{Q}(\sigma_i)\,,$$

which anticommute with D and γ^0.
Define

$$Q_i = iH^{-1}D^{(i)} = H^{-1}\mathcal{Q}(\sigma_i) = \mathcal{D}(\sigma_i)\,,$$

These operators form a representation of the algebra of Pauli matrices (quaternions)

$$Q_i\, Q_j = \delta_{ij} I + i\varepsilon_{ijk} Q_k\,.$$

Dirac-type operator from the Killing-Yano tensor f^Y is

$$D^Y = -\mathcal{Q}(\sigma_r) + \frac{2i}{\mu\sqrt{V}} \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix},$$

where $\sigma_r = \vec{\sigma}\cdot\vec{x}/r$ and $\lambda = \sigma_L + \mathbf{1_2} + \mu\sigma_r P_4$.
Define

$$Q^Y = HD^Y = HD^Y = \mathcal{D}(\sigma^Y)\,.$$

Conserved Runge-Lenz operator of the Dirac theory

$$\mathcal{K}_i = \frac{\mu}{4}\{Q^Y, Q_i\} + \frac{1}{2}(\mathcal{B} - P_4)Q_i - \mathcal{J}_i P_4\,,$$

where

$$\mathcal{B}^2 = {P_4}^2 - H^2 = \mathcal{D}(B^2).$$

Commutation relations for $\mathcal{J}_i$, $\mathcal{K}_j$:

$$\begin{aligned}
[\mathcal{J}_i,\,\mathcal{J}_j] &= i\varepsilon_{ijk}\mathcal{J}_k\,, \\
[\mathcal{J}_i,\,\mathcal{K}_j] &= i\varepsilon_{ijk}\mathcal{K}_k\,, \\
[\mathcal{K}_i,\,\mathcal{K}_j] &= i\varepsilon_{ijk}\mathcal{J}_k\mathcal{B}^2\,,
\end{aligned}$$

and commute with the operators Q_i

$$[\mathcal{J}_i,\,Q_j] = i\varepsilon_{ijk}Q_k\,, \qquad [\mathcal{K}_i,\,Q_j] = i\varepsilon_{ijk}Q_k\mathcal{B}\,.$$

This algebra does not close as a finite Lie algebra because of the factor $\mathcal{B}^2$.

The Casimir operators of the open algebra are

$$\mathcal{C}_1 = \vec{\mathcal{J}}^2\mathcal{B}^2 + \vec{\mathcal{K}}^2\,, \quad \mathcal{C}_2 = \vec{\mathcal{J}}\cdot\vec{\mathcal{K}}\,.$$

In addition, we can define a new Casimir-type operator

$$Q = \frac{\mu}{2}\,Q^Y + (\mathcal{B} - P_4)\mathcal{D}(\sigma_L + \mathbf{1}_2) = \mathcal{D}[\sigma_K + (\sigma_L + \mathbf{1}_2)B]\,,$$

which satisfies simple algebraic relations:

$$[Q, \mathcal{J}_i] = 0\,, \quad [Q, \mathcal{K}_i] = 0\,, \quad \{Q, Q_i\} = 2(\mathcal{K}_i + \mathcal{J}_i \mathcal{B})\,,$$

and the identity

$$Q^2 = \frac{\mu^2}{4}\,(P_4 - \mathcal{B})^4\,.$$

Introduce new Casimir operator

$$M = (N + \mu P_4)^2$$

that allows us to write

$$Q^2 = \{Q,\, Q\} = \mathcal{B}^2 M\,.$$

Infinite-dimensional superalgebra

How could be organized this very rich set of conserved Dirac operators?
Define of the *bosonic* operators

$$I_n = \mathcal{B}^n\,, \quad M_n = M\mathcal{B}^n\,, \quad J_n^i = \mathcal{J}_i\mathcal{B}^n\,, \quad K_n^i = \mathcal{K}_i\mathcal{B}^n\,,$$

and the supercharges of the *fermionic* sector

$$\mathcal{Q}_n = \mathcal{Q}\mathcal{B}^n\,, \qquad \mathcal{Q}_n^i = \mathcal{Q}_i\mathcal{B}^n\,,$$

for any $n = 0, 1, 2....$
Non-trivial commutators of the bosonic sector

$$\begin{aligned}
\left[J_n^i, J_m^j\right] &= i\varepsilon_{ijk}J_{n+m}^k\,,\\
\left[J_n^i, K_m^j\right] &= i\varepsilon_{ijk}K_{n+m}^k\,,\\
\left[K_n^i, K_m^j\right] &= i\varepsilon_{ijk}J_{n+m+2}^k\,,
\end{aligned}$$

Anticommutators of the fermionic sector,

$$
\begin{aligned}
\{Q_n^i, Q_m^j\} &= 2\delta_{ij} I_{n+m}\,, \\
\{Q_n, Q_m^i\} &= 2(K_{n+m}^i + J_{n+m+1}^i)\,, \\
\{Q_n, Q_m\} &= 2\, M_{m+n+2}\,.
\end{aligned}
$$

Commutations relations between the bosonic and fermionic operators

$$
\begin{aligned}
\left[Q_n, J_m^j\right] &= 0\,, \qquad\qquad [Q_n^i, J_m^j] = i\varepsilon_{ijk} Q_{n+m}^k\,, \\
\left[Q_n, K_m^j\right] &= 0\,, \qquad\qquad [Q_n^i, K_m^j] = i\varepsilon_{ijk} Q_{n+m+1}^k\,.
\end{aligned}
$$

Graded Kac-Moody loop superalgebra

J. Daboul, P. Slodowy, C. Daboul (1993, 1994).
I. I. Cotăescu, M. V. (2007)

Assign grades to each operator:

$$E_{2n} := \mathcal{B}^n\,, \quad F_{2n} := M\mathcal{B}^n\,, \quad A^i_{2n} := \mathcal{J}_i\mathcal{B}^n\,,$$
$$B^i_{2n+2} := \mathcal{K}_i\mathcal{B}^n\,, \quad G_{2n+2} := Q\mathcal{B}^n\,, \quad G^i_{2n} := Q_i\mathcal{B}^n\,.$$

Thus we achieve a graded loop superalgebra of the Kac-Moody type and the sum of the grades is conserved under (anti)commutations.
Countable set of operators

$$\{E_{2n}, F_{2n}, A^i_{2n}, B^i_{2n+2}, G_{2n+2}, G^i_{2n}\}\,, \quad n \in \mathbb{Z}\,,$$

generates a superalgebra.

Commutation relations of the bosonic sector,

$$
\begin{aligned}
\left[A^i_{2n},\, A^j_{2m}\right] &= i\varepsilon_{ijk} A^k_{2(n+m)}\,, \\
\left[A^i_{2n},\, B^j_{2m+2}\right] &= i\varepsilon_{ijk} B^k_{2(n+m+1)}\,, \\
\left[B^i_{2n+2},\, B^j_{2m+2}\right] &= i\varepsilon_{ijk} A^k_{2(n+m+2)}\,,
\end{aligned}
$$

Anticommutation relations of the fermionic sector

$$
\begin{aligned}
\left\{G^i_{2n}, G^j_{2m}\right\} &= 2\delta_{ij} E_{2(n+m)}\,, \\
\left\{G_{2n+2}, G^j_{2m}\right\} &= 2\left(A_{2(n+m+1)} + B_{2(n+m+1)}\right), \\
\{G_{2n+2}, G_{2m+2}\} &= 2\, F_{2(n+m+2)}\,,
\end{aligned}
$$

Commutation relations among both sectors,

$$
\begin{aligned}
&[A^i_{2n},\, G_{2m+2}] = 0\,, \qquad && [A^i_{2n},\, G^j_{2m}] = i\varepsilon_{ijk} C^k_{2(n+m)}\,, \\
&[B^i_{2n+2},\, G_{2m+2}] = 0\,, \qquad && [B^i_{2n+2},\, G^j_{2m}] = i\varepsilon_{ijk} C^k_{2(n+m+1)}\,.
\end{aligned}
$$

Gravitational anomalies

B. Carter (1977)
M. Cariglia (2004)
Classical motions a S-K tensor $K_{\mu\nu}$ generate a quadratic constant of motion

$$K = K_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,.$$

Quantum operator

$$\mathcal{K} = D_{\mu}K^{\mu\nu}D_{\nu}$$

Scalar Laplacian

$$\mathcal{H} = D_{\mu}D^{\mu}$$

Evaluate the commutator

$$\begin{aligned}[D_{\mu}D^{\mu}, \mathcal{K}] = \quad & 2K^{\mu\nu;\lambda}D_{(\mu}D_{\nu}D_{\lambda)} + 3K^{(\mu\nu;\lambda)}{}_{;\lambda}D_{(\mu}D_{\nu)} \\ & +\{\frac{1}{2}g_{\lambda\sigma}(K_{(\lambda\sigma;\mu);\nu} - K_{(\lambda\sigma;\nu);\mu}) - \frac{4}{3}K_{\lambda}{}^{[\mu}R^{\nu]\lambda}\}_{;\nu}D_{\mu}\end{aligned}$$

Hidden symmetry of the quantized system

$$[\mathcal{H}, \mathcal{K}] = -\frac{4}{3}\{K_\lambda{}^{[\mu} R^{\nu]\lambda}\}_{;\nu} D_\mu$$

On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S-K tensor $K_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

Integrability condition for K-Y tensors of valence $r = 2$

$$R_{\mu\nu[\sigma}{}^{\tau} f_{\rho]\tau} + R_{\sigma\rho[\mu}{}^{\tau} f_{\nu]\tau} = 0\,.$$

Contracting this integrability condition on the Riemann tensor

$$f^{\rho}{}_{(\mu} R_{\nu)\rho} = 0.$$

Suppose

$$K_{\mu\nu} = f_{\mu\rho} f_\nu{}^{\rho}.$$

Integrability condition becomes

$$K^{\rho}{}_{[\mu} R_{\nu]\rho} = 0.$$

The operators constructed from symmetric S-K tensors are in general a source of gravitational anomalies for scalar fields. However, when the S-K tensor admits a decomposition in terms of K-Y tensors the anomaly disappears. owing to the existence of the K-Y tensors.

Extended Taub-NUT spaces

T. Iwai, N. Katayama (1993, 1994)
Y. Miyake (1995)
M. Visinescu (2000)
I. I. Cotaescu, M. Visinescu (2001)
Extended Taub-NUT metric defined on $\mathbb{R}^4 - \{0\}$

$$ds_K^2 = f(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2) + g(r)(d\chi + \cos\theta \, d\varphi)^2$$

$f(r)$ and $g(r)$ are functions given, with constants a, b, c, d, by

$$f(r) = \frac{a + br}{r} \quad , \quad g(r) = \frac{ar + br^2}{1 + cr + dr^2} \, .$$

If one takes the constants

$$c = \frac{2b}{a}, d = \frac{b^2}{a^2}$$

the extended Taub-NUT metric becomes the original Euclidean Taub-NUT metric up to a constant factor.

Extended Taub-NUT space still admits a Runge-Lenz vector

$$\vec{K} = \vec{p} \times \vec{j} + \kappa \frac{\vec{r}}{r} .$$

with

$$\kappa = -a\,E + \frac{1}{2} c\,q^2$$

where the conserved energy E is

$$E = \frac{\vec{p}^{\,2}}{2f(r)} + \frac{q^2}{2g(r)} .$$

The components of $\vec{K}$ are S-K tensors.

Extended Taub-NUT is not Ricci flat.

There are no Killing-Yano tensors.

Extended Taub-NUT and gravitational anomalies

I. I. Cotăescu, S. Moroianu, M. V. (2005)

A direct evaluation shows that the commutator $[\mathcal{H}, \mathcal{K}]$ does not vanish implying the presence of the gravitational anomaly.

To illustrate for the third S-K $K_3^{\mu\nu}$ tensor in spherical coordinates

$$K_3^{rr} = -\frac{ar\cos\theta}{2(a+br)}\ , \ K_3^{r\theta} = K_3^{\theta r} = \frac{\sin\theta}{2}$$

$$K_3^{\theta\theta} = \frac{(a+2br)\cos\theta}{2r(a+br)}\ , \ K_3^{\varphi\varphi} = \frac{(a+2br)\cot\theta\csc\theta}{2r(a+br)}$$

$$K_3^{\varphi\chi} = K_3^{\chi\varphi} = -\frac{(2a+3br+br\cos(2\theta)\csc^2\theta}{4r(a+br)}$$

$$K_3^{\chi\chi} = \frac{(a-adr^2+br(2+cr)+(a+2br))\cot^2\theta)\cos\theta}{2r(a+br)}\,.$$

Just to exemplify, we write down from the commutator $[\mathcal{H}, \mathcal{K}_3]$ the function which multiplies the covariant derivative D_r

$$\frac{3r\cos\theta}{4(a+br)^3(1+cr+dr^2)^2} \cdot$$
$$\{-2bd(2ad-bc)r^3 +$$
$$[3bd(2b-ac)-(ad+bc)(2ad-bc)]r^2 +$$
$$2(ad+bc)(2b-ac)r + a(2ad-bc) + (b+ac)(2b-ac)\}$$

As it is expected there is no gravitational anomaly for the standard Euclidean Taub-NUT metric ($c = \frac{2b}{a}, d = \frac{b^2}{a^2}$) .

Index formulas and axial anomalies

M. F. Atiyah, V. K. Patodi, I. M. Singer (1975, 1976)
H. Römer, B. Schroer (1977)
C. N. Pope (1978)
T. Eguchi, P. B. Gilkey, A. J. Hanson (1978)
K. Peeters, A. Waldron (1999)
I. I. Cotăescu, S. Moroianu, M. V. (2005)
S. Moroianu, M. V. (2006)

Let (M, g) be a closed Riemannian spin manifold of odd dimension, Σ the spinor bundle and D the (self-adjoint) Dirac operator on M. Let

$$\Pi^{\pm} : \mathcal{C}^{\infty}(M, \Sigma) \to \mathcal{C}^{\infty}(M, \Sigma)$$

be the spectral projections associated to D and the intervals $[0, \infty)$, respectively $(-\infty, 0]$. If ϕ_T is an eigenspinor of D of eigenvalue T, then

$$\Pi^{+}(\phi_T) = \begin{cases} \phi_T & \text{if } T \geq 0; \\ 0 & \text{otherwise;} \end{cases} \qquad \Pi^{-}(\phi_T) = \begin{cases} \phi_T & \text{if } T \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let now g^X be a Riemannian metric on the cylinder $X := [l_1, l_2] \times M$. Endow X with the product orientation, so that $\{l_1\} \times M$ is negatively oriented and $\{l_2\} \times M$ is positively oriented inside X. Let D^+ be the chiral Dirac operator on X. For each $t \in [l_1, l_2]$ let g_t be the metric on M obtained by restricting g^X to $\{t\} \times M$. We denote by Σ_t the spinor bundle over (M, g_t) and by D_t, $\Pi_t^{\pm}$ the Dirac operator and the spectral projections with respect to the metric g_t.
There exist canonical identifications of the spinor bundle Σ_t with $\Sigma^{\pm}(X)_{|\{t\}\times M}$. Denote by ϕ_t the restriction of a positive spinor from X to $\{t\} \times M$.

Theorem 1 *Let $X = [l_1, l_2] \times M$ be a product spin manifold with a smooth metric g^X as above. Set*

$$\mathcal{C}^\infty(X, \Sigma^+, \Pi^+) := \{\phi \in \mathcal{C}^\infty(X, \Sigma^+); \Pi_{l_1}^- \phi_{l_1} = 0, \Pi_{l_2}^+ \phi_{l_2} = 0\}.$$

Then the operator

$$D^+ : \mathcal{C}^\infty(X, \Sigma^+, \Pi^+) \to \mathcal{C}^\infty(X, \Sigma^-)$$

is Fredholm, of index equal to the spectral flow of the pair (D_{l_1}, D_{l_2}).

Berger introduced a family of Riemannian metrics on the 3-sphere as follows: The Hopf fibration $h : S^3 \to S^2$ defines a vertical subbundle V in TS^3. Let $H \subset TS^3$ be the orthogonal complement with respect to the standard metric g_{S^3}. Then h becomes a Riemannian submersion when we endow S^3 with its standard metric, and S^2 with 4 times its standard metric. Let g_H, g_V denote the restriction of g_{S^3} to the horizontal, respectively the vertical bundle.

For each constant $\lambda > 0$ the Berger metric g_λ on S^3 is defined by the formula

$$g_\lambda := g_H + \lambda^2 g_V .$$

Lemma 2 *For $\lambda < 2$, D_λ has no harmonic spinors.*

Proof: It is easy to compute the scalar curvature of g_λ. Namely, $\kappa(g_\lambda)$ is constant on S^3, $\kappa(g_\lambda) = (4 - \lambda^2)/12$. In particular $\kappa(g_\lambda)$ is positive for $\lambda < 2$. Lichnerowicz's formula proves then that $\ker D_\lambda = 0$. ■

Theorem 3 *Let*

$$\Lambda(\lambda) := \{(p,q) \in \mathbb{N}^{*2}; \lambda^2 = 2\sqrt{(p-q)^2 + 4\lambda^2 pq}\}.$$

Then

$$\dim \ker(D_\lambda) = N(\lambda) := \sum_{(p,q)\in\Lambda(\lambda)} p + q.$$

If $N(\lambda) > 0$ *there exists* $\epsilon > 0$ *such that for* $|t - \lambda| < \epsilon$*, the "small" eigenvalues of* D_t *are given by families*

$$T(t,p,q) := \frac{t}{2} - \sqrt{\frac{(p-q)^2}{t^2} + 4pq}, \quad (p,q) \in \Lambda(\lambda)$$

with multiplicity $p + q$*.*

In particular, the first harmonic spinors appear for $\lambda = 4$ where the kernel of D_4 is two-dimensional. Moreover, the set of those $\lambda \in (0, \infty)$ for which $N(\lambda) \neq 0$ is discrete. For $l > 0$ set

$$S(l) := \sum_{\lambda \leq l} N(\lambda).$$

Corollary 4 *The spectral flow of the family $\{D_t\}_{t\in[l_1,l_2]}$ of Berger Dirac operators equals $S(l_2) - S(l_1)$.*

Proof: By differentiating $T(t,p,q)$ we see that the function $t \to T(t,p,q)$ is strictly increasing, so the spectral flow of the family $\{D_t\}$ across λ is precisely $N(\lambda)$. ■

For the extended Taub-NUT metric ds_K^2 on $\mathbb{R}^4 \setminus \{0\} \simeq (0,\infty) \times S^3$ in terms of the Berger metrics

$$ds_K^2 = (ar + br^2)\left(\frac{dr^2}{r^2} + 4g_{\lambda(r)}\right),$$

where

$$\lambda(r) := \frac{1}{\sqrt{1 + cr + dr^2}}$$

Axial anomalies translate to Dirac operators with non-vanishing index. We are interested in the chiral Dirac operator on a annular piece of $\mathbb{R}^4 \setminus \{0\}$. First set $X_{l_1,l_2} := [l_1,l_2] \times S^3 \subset \mathbb{R}^4 \setminus \{0\}$ with the induced extended Taub-NUT metric.

Theorem 5 *The index of D^+ over (X_{l_1,l_2}, ds^2_K) with the APS boundary condition is*

$$\text{index}(D^+) = S(\lambda(l_2)) - S(\lambda(l_1))$$

Proof: By Theorem 1 the index is equal to the spectral flow of the pair of boundary Dirac operators. Now the metrics on the boundary spheres are constant multiples of the Berger metrics $g_{\lambda(l_1)}$, respectively $g_{\lambda(l_2)}$. The spectral flow of a path of conformal metrics (even with non-constant conformal factor) vanishes by the conformal invariance of the space of harmonic spinors . Thus the spectral flow can be computed using the pair of metrics $g_{\lambda(l_1)}$ and $g_{\lambda(l_2)}$. The conclusion follows from Corollary 4. ■

Corollary 6 *If $c > -\frac{\sqrt{15d}}{2}$ then the extended Taub-NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).*

Proof: The hypothesis implies that $\lambda(r) < 4$ for all $r > 0$. From the remark following Theorem 3 we see that $S(\lambda(l_1)) = S(\lambda(l_2)) = 0$. ■

Summary

- K-Y tensors and supersymmetries
- Covariantly constant K-Y tensors and Dirac type operators
- Non-covariantly constant K-Y tensors and hidden symmetries
- K-Y tensors and anomalies

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