On supergroups with odd Clifford parameters and Non-anticommutative (NAC) supersymmetry

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CBPF

Talk at BW2007 - Kladovo

based on arXiv: 0705.4007 [hep-th]

- Extension of Berezin
- Connection with Bender's PT-symmetric (pseudo-hermitian) Hamiltonians

NAC - SUPERSYMMETRY

Schwarz - Van Nieuwenhuizen '82

Ooguri - Vafa 2003 Seiberg, Berkovits

bckgr. of superstring.

Seiberg JHEP 0306 (2003) NAC SUSY in 4D EUCLIDEAN SPACE

Moyal $*$-product (coming from NC-theory)

"Cliffordization of the superspace coordinates"

$$\{\theta_\alpha, \theta_\beta\} = C_{\alpha\beta}$$

Lower dimensional NAC susy theory:

in 2D: Chandrasekhar - Kumar '04
Hatanaka - Ketov - Kobayashi - Sasaki '05
Álvarez - Gaumé - Vazquez - Mozo '05

in 3D: ....

in 1D: Aldrovandi - Schaposnik (2006)

Up

Bottom

approach: deformation

Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer '78

Example: Hirshfeld-Henselder '02:

Grassmann algebra $\theta_i\theta_j + \theta_j\theta_i = 0$ $*$-deformed

$$\Downarrow$$

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\eta_{ij}$$

TOP

DOWN

Framework: limit procedure:

example: Inonü-Wigner contraction

$c \to \infty$

....

Basic idea:

$$\text{Superalgebra } \mathfrak{g} \xrightarrow{\text{"exp"}} \text{Supergroup } G$$

$\mathbb{Z}_2$ grading

even generators $t_i \in \mathfrak{g}$
odd generators $\tau_j \in \mathfrak{g}$

$t_i \longleftrightarrow x_i$ bosonic parameters

$\tau_i \longleftrightarrow \theta_j$ Grassmann parameters

$$g \approx \exp x_i t_i + \theta_j \tau_j$$

Relaxing the condition for θ's to be Grassmann.

Promoting θ's to be Clifford

$$\theta_i \theta_j + \theta_j \theta_i = 2\eta_{ij}$$

$Cl(p,q,r)$

$$\begin{pmatrix} \overbrace{+ \ \ddots \ +}^{p} & & \\ & \overbrace{- \ \ddots \ -}^{q} & \\ & & \overbrace{0 \ \ddots \ 0}^{r} \end{pmatrix}$$

Simplest scenario:

Superalgebra of the N-extended 1D S.Q.M.

$$\{Q_i, Q_j\} = 2\delta_{ij} H$$
$$[H, Q_i] = 0$$

$i,j = 1,\ldots,N$

Take $N = 1,2$.

Conjugation $(ab)^\dagger = b^\dagger a^\dagger$

$(a^\dagger)^\dagger = a$

$N=1$ unitary supergroup element

$$g = e^{-iHt} e^{\lambda\theta Q} \qquad \theta^\dagger = \theta$$

$[H]=1 \qquad [t]=-1$

$[Q]=\frac{1}{2} \qquad [\theta_\lambda]=-\frac{1}{2} \qquad \theta_\lambda=\lambda\theta \qquad [\theta]=-\frac{1}{2}.$

$$\theta^2=\frac{\epsilon}{M} \qquad \epsilon=+1,0,-1$$

Three cases:

$\epsilon=0) \longrightarrow \; g=e^{-iHt}(1+\lambda\theta Q)$ rational

$\epsilon=1) \longrightarrow \; g=e^{-iHt}(\cos X+I\sin X)$ trigonometric

$$I=\theta Q\sqrt{\frac{M}{H}} \qquad I^2=-1$$

$\epsilon=-1) \longrightarrow \; g=e^{-iHt}(\cosh X+J\sinh X)$ hyperbolic

$$J=\theta Q\sqrt{\frac{M}{H}} \qquad J^2=+1$$

$$X=\sqrt{\frac{\lambda^2 H}{M}}$$

Ordinary Grassmann case recovered in the limit $M\to\infty$

$N=2$ SQM: $\theta_i^2 = \frac{\epsilon_i}{M}$

Three cases of interest: $\epsilon = \epsilon_1 \epsilon_2$

$\epsilon = 0$ $Cl(1,0,1)$ $Cl(0,1,1)$ $Cl(0,0,2)$

$\epsilon = +1$ $Cl(2,0)$ or $Cl(0,2)$ "EUCLIDEAN"

$\epsilon = -1$ $Cl(1,1)$ "LORENTZIAN"

Both θ_i's are Clifford-deformed.

Berezin-like calculus for odd-Clifford variables

grassmann:
$$\partial_\theta \theta = 1 \qquad \int_\theta = \partial_\theta$$
$$\partial_\theta 1 = 0$$

Let us recover Berezin in the $M \to \infty$ limit $\theta^2 = \frac{\epsilon}{M}$

$$\partial_\theta \theta = 1$$
$$\partial_\theta 1 = 0$$

Extended by graded Leibniz rule:
$$\partial_\theta (f_1 f_2) = \partial_\theta f_1 \cdot f_2 + (-)^{\deg f_1} f_1 (\partial_\theta f_2)$$

$$\Downarrow$$
$$\partial_\theta \theta^{2k} = 0$$
$$\partial_\theta \theta^{2k+1} = \theta^{2k}$$

In order that the integral of a total derivative should be vanishing:
$$\underline{\int_\theta \theta^{2k} = 0}$$

Moreover: $\int_\theta \theta^{2k+1} = \theta^{2k}$ $\qquad \underline{\int_\theta = \partial_\theta}$

Odd-Clifford valued superfield Φ

$$\Phi(\theta) = \varphi + i\psi\theta$$

wher $\varphi = \sum_{n=0}^{+\infty} \frac{\varphi_n}{M^n}$ φ_n subcomponents of

mass-dimension $[\varphi_n] = d+n$

The Grassmann case is recovered by φ_0 when taking $M \to \infty$

Trivial extension of the calculus to N odd-Clifford variables.

Fermionic covariant derivative (Salam-Strathdee)

$$Qg = Q_L g$$
$$gQ = Q_R g$$

$$\left[\begin{aligned} \{Q_L, Q_L\} &= -H \\ \{Q_L, Q_R\} &= 0 \\ \{Q_R, Q_R\} &= H \end{aligned}\right.$$

$Q_L \equiv D$

In Grassmann
$$Q_L = \frac{1}{\lambda}\partial_\theta - i\lambda\theta\partial_t$$
$$Q_R = \frac{1}{\lambda}\partial_\theta + i\lambda\theta\partial_t \qquad \text{set } \lambda = 1$$

In odd-Clifford:
$$Q_L = \partial_\theta\partial_\lambda - i\theta\partial_t \int_\lambda + \frac{i\epsilon}{M}\partial_\theta\partial_t\int_\lambda$$
$$Q_R = \partial_\theta\partial_\lambda + i\theta\partial_t \int_\lambda - i\frac{\epsilon}{M}\partial_\theta\partial_t\int_\lambda$$

Applied to exp(λ) $\Downarrow$

$$\left[\begin{aligned} D &= \partial_\theta - i\theta\partial_t + i\frac{\epsilon}{M}\partial_\theta\partial_t \\ Q &= \partial_\theta + i\theta\partial_t - i\frac{\epsilon}{M}\partial_\theta\partial_t \end{aligned}\right.$$

The component fields have the same transformation properties both in Grassmann and odd-Clifford cases:

$$\delta_\epsilon \varphi = -i\,\epsilon \psi$$
$$\delta_\epsilon \psi = \epsilon\,\partial_t \varphi$$

$N=2$ extension:

$$D_j = \partial_{\theta_j} - i\theta_j \partial_t + i\,\frac{\epsilon_j}{M}\,\partial_{\theta_j}\partial_t$$
$$Q_j = \partial_{\theta_j} + i\theta_j \partial_t - i\,\frac{\epsilon_j}{M}\,\partial_{\theta_j}\partial_t$$

$$\{D_i, D_j\} = -\delta_{ij} H$$
$$\{D_i, Q_j\} = 0$$
$$\{Q_i, Q_j\} = \delta_{ij} H$$

(anti) symmetrized $*$-multiplication

$$A_1 * A_2 = \frac{1}{2}\, A_1 A_2 + (-)^{d_{A_1} \cdot d_{A_2}} A_2 A_1$$

$$\underset{\mathbb{R}}{B} * \underset{\mathbb{R}}{B} = \underset{\mathbb{R}}{B}$$

$$\underset{\mathbb{R}}{B} * \underset{\mathbb{R}}{F} = \underset{\mathbb{R}}{F}$$

$$\underset{\mathbb{R}}{F} * \underset{\mathbb{R}}{F} = \underset{\mathrm{Im.}}{B}$$

$N=2$ $(1,2,1)$ real multiplet

$$\phi = \varphi + i\,\psi_1 \theta_1 + i\,\psi_2 \theta_2 + i\,\theta_1 \theta_2 f$$

$(2,2)$ chiral multiplet

Complexified field Υ s.t. $(D_1 + i D_2)\Upsilon = 0$

For $N=2$ the graded Leibniz rule is broken for $\varepsilon \neq 0$

$$\Delta_{\dot{1}}(A,B) = -i\frac{\epsilon}{M^2}\left(f_B\dot{\psi}_{2A} + f_A\dot{\psi}_{2B} + (f_A\dot{f}_B + \dot{f}_A f_B)\theta_1\right)$$

breaking of graded Leibniz.

The $\hat{*}$ multiplication preserves Leibniz.

$$A\,\hat{*}\,B = A * B + \frac{\epsilon}{M^2}\,\partial_{\theta_1}\partial_{\theta_2}A\cdot\partial_{\theta_1}\partial_{\theta_2}B$$

The integration over bilinear terms produce the same results as in the Grassmann case. Trilinear terms and beyond.

Simplest non-trivial example ϕ (1,2,1)-real multiplet.

N=2 S.Q.M. $S_{kin} = \frac{1}{2m} \iint (D_1\phi \star D_2\phi)$

$\hat{\star}$

Trilinear potential (real)

$$S_{pot} = i\frac{w}{2} \iint c_1\, \phi \star \phi \star \phi + c_2 \phi \star \phi + c_3\phi$$

Set $c_1 = \frac{1}{6}$ $\phi \to \phi' = \phi + c$ we end up with a single real parameter α:

$$\frac{1}{6}\phi^{\star 3} + \alpha\phi$$

For $\star$ $S_{kin} + S_{pot} \equiv K - V$

$$K = \frac{1}{2m}\left(\dot{\varphi}^2 - i\dot{\psi}_1\psi_1 - i\dot{\psi}_2\psi_2\right)$$

$$V = -\left(\frac{1}{2m}f^2 + \frac{1}{2}\varphi^2 f - i\varphi\psi_1\psi_2 + \frac{1}{6}\frac{\epsilon}{M^2}f^3 + \alpha f\right)$$

Three cases

i) $\epsilon = 0$ Grassmann.

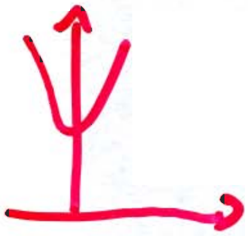
Solving the E.O.M. of the auxiliary field (linear) we get the potential (purely bosonic case, setting $\psi_1 \equiv \psi_2 \equiv 0$

$$V(\varphi) = \frac{1}{8}\left(\varphi^2 + 2\alpha\right)^2$$

φ^4 potential.

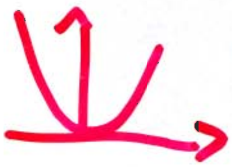
Two invariances: susy and Z_2 $\varphi \to -\varphi$ symmetry.

A) $\alpha > 0$



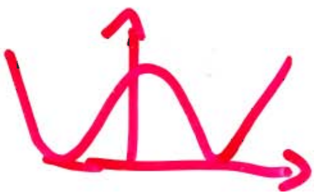
Z_2 exact
SUSY BROKEN

B) $\alpha = 0$



Z_2 and susy exact

C) $\alpha < 0$



Z_2 Broken (spontaneously)
SUSY exact.

ii) $\epsilon = -1$ "Lorentzian" case:

$$f = \begin{cases} m(1+x) \\ m(1-x) \end{cases} \qquad x = \sqrt{1+2\alpha+y^2}$$

two branches

$$\frac{V_\pm}{m} = \pm \frac{1}{3} x^3 - \frac{1}{2} x^2 + \frac{1}{6}$$

For $\alpha \geq -\frac{1}{2}$, x is always real. The branches have to be chosen s.t. V is bounded below

$$\frac{V}{m} = \frac{1}{3} |x^3| - \frac{1}{2} x^2 + \frac{1}{6}$$

A) $\alpha > 0$ $\quad Z_2$ exact, susy broken

B) $\alpha = 0$ $\quad Z_2$ and susy exact

C) $-\frac{1}{2} \leq \alpha < 0$: DEFORMED VERSION OF THE "MEXICAN HAT" SUSY EXACT, Z_2 BROKEN.

What about $\alpha < -\frac{1}{2}$ and $\epsilon = +1$?

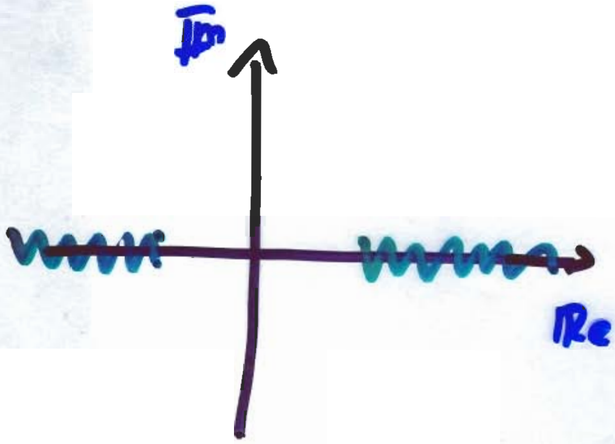
For $\epsilon = +1$ $\qquad x = \sqrt{1 - 2\alpha - \varphi^2}$

The potential $\pm\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}$ is universal.

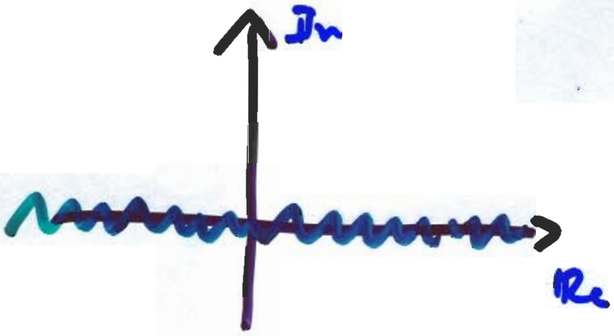
What changes is the domain of x: we have

6 cases:

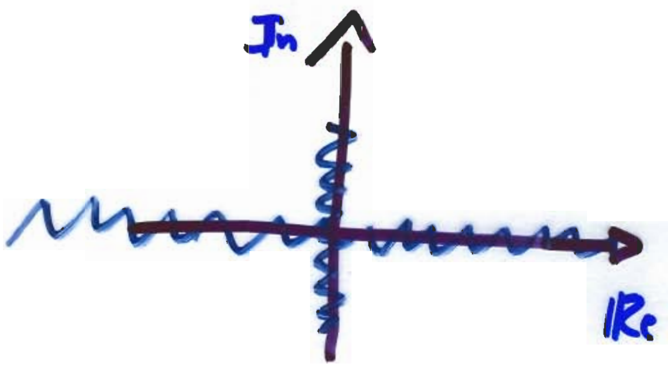
$\epsilon = -1 \quad \alpha > 0$



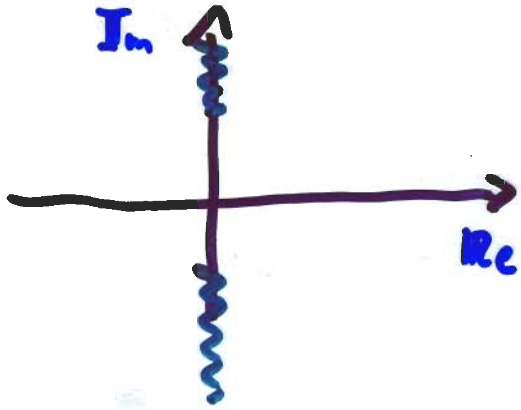
$\epsilon = -1 \quad \alpha = 0$



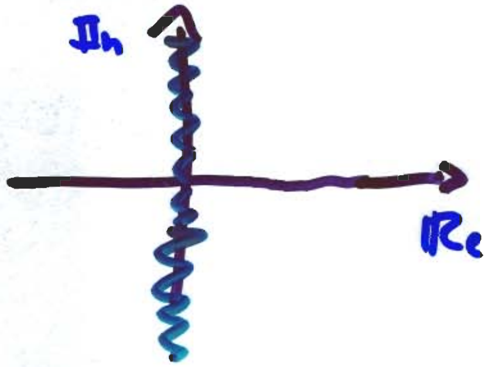
$\epsilon = -1 \quad \alpha < 0$



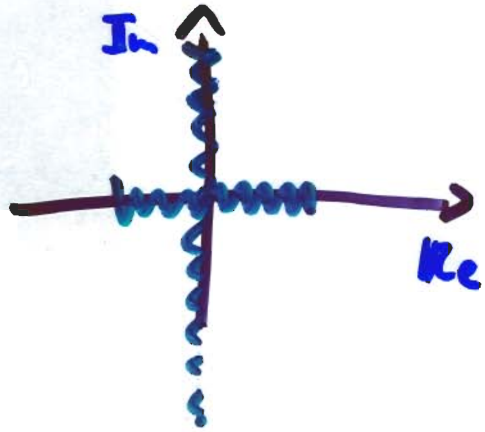
$\epsilon = +1 \quad \alpha > \frac{1}{2}$



$\epsilon = +1 \quad \alpha = \frac{1}{2}$



$\epsilon = +1 \quad \alpha < \frac{1}{2}$



For both $\epsilon = \pm 1$ and any α, the action written in terms of the auxiliary field is real. The imaginary unit is introduced when solving the E.O.M. for f.

Consider the Euclidean case with $\underline{\alpha = \frac{1}{2}}$

$$\underline{x = i\varphi}$$

$$S = \int dt \left(\frac{1}{2}\dot{\varphi}^2 + \frac{i}{3}\varphi^3 - \frac{1}{2}\varphi^2 - \frac{1}{6} \right)$$

$m \equiv 1$

$$\Downarrow$$

$$\underline{H = \frac{1}{2}p^2 - \frac{i}{3}\varphi^3 + \frac{1}{2}\varphi^2 + \frac{1}{6}}$$

(PT-symmetric hamiltonian with a mass-term)

Bender-Boettcher pseudo-hermitian hamiltonians

$$H = \frac{1}{2}p^2 - \frac{i}{3}q^3$$

with real spectrum:

PT-symmetric: $\mathcal{P}: q \mapsto -q \quad p \mapsto -p$

Bender's approach $\mathcal{T}: q \mapsto q \quad p \mapsto -p \quad i \mapsto -i$

Mostafazadeh: conjugation transformation

$$\tilde{H} = e^R H e^{-R}$$

self-adjoint $\tilde{H}$.

Crypto-reality (Ivanov-Smilga) '07 NAC-surv.

Drinfeld twist deformation of various supersymmetry algebras:

Kobayashi - Sasaki '05
Zupnik '05
Ihl - Sämann '06

$\Delta(Q_i) = Q_i \otimes 1 + 1 \otimes Q_i + \dots$ deformed coproduct

$\uparrow$ supersymmetry generators

$$\delta_\epsilon(\phi_1 \cdot \phi_2) \neq \delta_\epsilon \phi_1 \cdot \phi_2 + \phi_1 \delta_\epsilon \phi_2$$

Physical interpretation of the coproduct for tensoring of multiparticle states.

Example Jordanian deformation of $sl(2)$ (Ogievetsky)

$\Delta_0(H) = H \otimes 1 + 1 \otimes H \Rightarrow$ Addition of energy $E_{1+2} = E_1 + E_2$

$\Delta_J(H) \qquad E_{1+2} \neq E_1 + E_2$

Extension to higher-dimensions:

immediate

real spinors $\{\theta_\alpha, \theta_\beta\} = C_{\alpha\beta}$

for space-time carrying a symmetric charge conjugation matrix $C_{\alpha\beta}$.

For Dirac spinors $\{\theta_A, \theta_B^+\} = A_{AB}$

hermitian matrix A.

In $D=1$ construction for higher N (example $N=4$).

For Superconformal theories, what is the role of the mass-scale M, ...