# Central charge contribution to non-commutativity

B. Nikolić and B. Sazdović
Institute of Physics Belgrade

# BW2007

*III Southeastern European Workshop*
*Challenges Beyond the Standard Model*
September 2 - 9, 2007
Kladovo, Serbia

# Outline of the talk

- Basic of string theory
- Variational principle, boundary conditions and $Dp$-branes
- Definition of the model
- Conformal anomaly and space-time field equations
- Inclusion of Lioville term and $Dp$-brane properties
- Conclusions

# Basic of string theory

- Strings are objects with one spatial dimension.
- During motion string sweeps a two-dimensional surface called world-sheet.
- The world-sheet is parameterized by two parameters: one time-like $\tau$ and one space-like $\sigma$, $\sigma \in [0\,,\pi]$.
- Strings occur in two toplogies: closed, which do not have endpoints, and open strings, where contribution of boundary conditions is nontrivial.

# Variational principle and boundary conditions

- Let action $S$ depends on the space-time coordinates $x^\mu, (\mu = 0, 1, \ldots, D)$ and their derivatives with respect to $\tau$ and $\sigma$, $\dot{x}^\mu$ and $x'^\mu$, respectively. A variation yields

$$\delta S = \int d\tau d\sigma (\frac{\partial \mathcal{L}}{\partial x^\mu} - \partial_\tau \pi_\mu - \partial_\sigma \gamma_\mu^{(0)}) \delta x^\mu + \int d\tau \gamma_\mu^{(0)} \delta x^\mu \Big|_0^\pi , \tag{1}$$

  where $\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$ and $\gamma_\mu^{(0)} = \frac{\partial \mathcal{L}}{\partial x'^\mu}$.
- The first term gives Euler-Lagrangian equations of motion, while vanishing of the second term gives boundary conditions.
- The closed strings satisfy boundary conditions automtically, while in the case of the open ones we have to examine their contribution to the string dynamics.

# Sorts of boundary conditions

- Arbitrary coordinate variations $\delta x^\mu$ at string endpoints gives Neumann boundary conditions

$$\gamma_\mu^{(0)}\big|_0 = \gamma_\mu^{(0)}\big|_\pi = 0\,. \qquad (2)$$

- Fixed coordinates at the string endpoints

$$\delta x^\mu\big|_0 = \delta x^\mu\big|_\pi = 0\,, \qquad (3)$$

gives Dirichlet boundary conditions.

# $Dp$-branes

- $Dp$-branes are $p+1$-dimensional objects with $p$ spatial dimensions which satisfy Dirichlet boundary conditions.

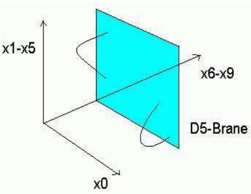


Figure: Example of D5-brane

- In $D$-dimensional space-time for coordinates $x^i \, (i = 0 \, , 1 \, , 2 \, , \ldots \, , p)$ we choose Neumann boundary conditions, and for the rest ones $x^a \, (a = p+1, \ldots, D)$ Dirichlet boundary conditions, so that $G_{\mu\nu} = 0 \, (\mu = i \, , \nu = a)$.

# Definition of the model

## Action

- Let us introduce the action which desribes the string dynamics in the presence of metric $G_{\mu\nu}(x)$, antisymmetric Kalb-Ramond field $B_{\mu\nu}(x)$ and dilaton field $\Phi(x)$

$$S = \kappa \int_{\Sigma} d^2\xi \sqrt{-g} \left\{ \left[ \frac{1}{2} g^{\alpha\beta} G_{\mu\nu} + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi R^{(2)} \right\} , \tag{4}$$

where $\xi^\alpha = (\tau , \sigma)$ parameterizes the world-sheet $\Sigma$ with metric $g_{\alpha\beta}$. Symbol $R^{(2)}$ denotes scalar curvature corresponding to the metric $g_{\alpha\beta}$.

# Quantum world-sheet conformal invariance and space-time field equations

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0\,, \tag{5}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} = 0\,, \tag{6}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = 0\,, \tag{7}$$

where $R_{\mu\nu}$, $D_\mu$ and $R$ are Ricci tensor, covariant derivative and scalar curvature with respect to the metric $G_{\mu\nu}$, $B_{\mu\rho\sigma} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is field strength for the field $B_{\mu\nu}$ and the vector $a_\mu = \partial_\mu \Phi$ is gradient of dilaton field.

- One particular solution of these equations is

$$G_{\mu\nu}(x) = G_{\mu\nu} = const\,, B_{\mu\nu}(x) = B_{\mu\nu} = const\,, \tag{8}$$

$$\Phi(x) = \Phi_0 + a_\mu x^\mu\,, (a_\mu = const)\,. \tag{9}$$

# Quantum conformal invariance - Lioville term

- If $\beta^G_{\mu\nu} = 0$ and $\beta^B_{\mu\nu} = 0 \Longrightarrow \beta^\Phi = c$, where $c$ is a constant. (C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Nucl. Phys.* **B 262** (1985) 593)
- For $G_{\mu\nu} = const$, $B_{\mu\nu} = const$ and $\Phi = \Phi_0 + a_\mu x^\mu$ we have

$$\beta^\Phi = 2\pi\kappa \frac{D-26}{6} + 4a^2 \equiv c \,. \tag{10}$$

- The nonlinear sigma model (4) becomes conformal field theory characterized by Virasoro algebra with central charge $c$.
- The remaining anomaly can be cancelled by adding Liouville term to the action (4)

$$S_L = -\frac{\beta^\Phi}{2(4\pi)^2\kappa} \int_\Sigma d^2\xi \sqrt{-g} R^{(2)} \frac{1}{\Delta} R^{(2)} \,, \quad \Delta = g^{\alpha\beta} \nabla_\alpha \partial_\beta \,, \tag{11}$$

where $\nabla_\alpha$ is the covariant derivative with respect to $g_{\alpha\beta}$.

# Quantum conformal invariance - full action

- Osillations in $x^a$ directions decouple from the rest. We use conformal gauge, $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$. Adding Liouville term, which is quadratic in $F$, and changing variable $F \to {}^\star F = F + \frac{\alpha}{2} a_i x^i$, we cancel term linear in $F$

$$S = \kappa \int_\Sigma d^2\xi \left[ \left( \frac{1}{2}\eta^{\alpha\beta}\, {}^\star G_{ij} + \epsilon^{\alpha\beta} B_{ij} \right) \partial_\alpha x^i \partial_\beta x^j + \frac{2}{\alpha}\eta^{\alpha\beta}\partial_\alpha {}^\star F \partial_\beta {}^\star F \right], \tag{12}$$

where

$$ {}^\star G_{ij} = G_{ij} - \alpha a_i a_j\,, \qquad \left( \frac{1}{\alpha} = \frac{\beta^\Phi}{(4\pi\kappa)^2} \right) \tag{13}$$

depends on the central charge $c$.

- The field ${}^\star F$ decouples, and the rest part of the action has a dilaton free form up to the change $G_{ij} \to {}^\star G_{ij}$, where ${}^\star G_{ij}$ can be singular for some choices of background fields.
- For $x^i$ and ${}^\star F$ we choose Neumann boundary conditions, which will be treated as canonical constraints.

**Case (1) -** $A \equiv 1 - \alpha a^2 \neq 0$ **and** $\tilde{A} \equiv 1 - \alpha \tilde{a}^2 \neq 0$

# Hamiltonian and currents

- From $\det {}^\star G_{ij} = A \det G_{ij}\,, (\det G_{ij} \neq 0)$ follows that redefined metric ${}^\star G_{ij}$ is nonsingular. Because ${}^\star F$ decouples, this case is equivalent to the dilaton free case.
- Canonical Hamiltonian is of the form

$$H_c = \int d\sigma \mathcal{H}_c\,, \quad \mathcal{H}_c = T_- - T_+\,,$$
$$T_\pm = \mp \frac{1}{4\kappa}\left[({}^\star G^{-1})^{ij}\, {}^\star j_{\pm i}\, {}^\star j_{\pm j} + \frac{\alpha}{4}\, {}^\star j_{\pm(F)}\, {}^\star j_{\pm(F)}\right] \quad (14)$$

where the currents are defined as

$${}^\star j_{\pm i} = \pi_i + 2\kappa\, {}^\star\Pi_{\pm ij} x'^j\,, \quad {}^\star j_{\pm(F)} = \pi \pm \frac{4\kappa}{\alpha} {}^\star F'\,, \quad (15)$$

and $({}^\star G^{-1})^{ij} = G^{ij} + \frac{\alpha}{1-\alpha a^2} a^i a^j$ and ${}^\star\Pi_{\pm ij} = B_{ij} \pm \frac{1}{2} {}^\star G_{ij}$. The canonical momenta are denoted by $\pi_i$ and $\pi$.

# Boundary conditions

- Boundary conditions in terms of currents

$$\gamma_i^{(0)} = ({}^\star\Pi_+ \, {}^\star G^{-1})_i{}^j \, {}^\star j_{-j} + ({}^\star\Pi_- \, {}^\star G^{-1})_i{}^j \, {}^\star j_{+j} \,, \tag{16}$$

$$\gamma^{(0)} = \frac{1}{2}\left[{}^\star j_{-(F)} - {}^\star j_{+(F)}\right] \,. \tag{17}$$

- Examing the consistency of the constraints at $\sigma = 0$, using Taylor expansion, we obtain

$$\begin{aligned} \Gamma_i(\sigma) &= ({}^\star\Pi_+ \, {}^\star G^{-1})_i{}^j \, {}^\star j_{-j}(\sigma) + ({}^\star\Pi_- \, {}^\star G^{-1})_i{}^j \, {}^\star j_{+j}(-\sigma) \,, \\ \Gamma(\sigma) &= \frac{1}{2}\left[{}^\star j_{-(F)}(\sigma) - {}^\star j_{+(F)}(-\sigma)\right] \,. \end{aligned} \tag{18}$$

- In the same way we obtain corresponding expressions at $\sigma = \pi$. The periodicity of canonical variables solves the boundary conditions at $\sigma = \pi$ and we consider only (18).

# Algebra of constraints

- Algebra of the constraints $\chi_A = (\Gamma_i, \Gamma)$ is

$$\{\chi_A(\sigma), \chi_B(\overline{\sigma})\} = -\kappa M_{AB}\delta' \,, \qquad M_{AB} = \begin{pmatrix} {}^\star G^{eff}_{ij} & 0 \\ 0 & \frac{4}{\alpha} \end{pmatrix}, \tag{19}$$

  where

$$ {}^\star G^{eff}_{ij} = {}^\star G_{ij} - 4(B\, {}^\star G^{-1} B)_{ij} \,. \tag{20}$$

- From

$$\det {}^\star G^{eff}_{ij} = \frac{\tilde{A}^2}{A} \det G^{eff}_{ij} \,, \tag{21}$$

  follows that all constraints $\chi_A$ are of the second class for $\tilde{A} \neq 0$.

# Solution of constraints

- Solving $\Gamma_i = 0$ and $\Gamma = 0$, we get

$$x^i(\sigma) = q^i(\sigma) - 2\,{}^\star\Theta^{ij} \int_0^\sigma d\sigma_1 p_j(\sigma_1)\,, \quad \pi_i = p_i\,, \tag{22}$$

$${}^\star F = {}^\star f\,, \quad \pi = p\,, \tag{23}$$

where

$$q^i(\sigma) = \frac{1}{2}\left[x^i(\sigma) + x^i(-\sigma)\right]\,, \quad p_i(\sigma) = \frac{1}{2}\left[\pi_i(\sigma) + \pi_i(-\sigma)\right]\,, \tag{24}$$

and similar for ${}^\star f$ and $p$.

- Antisymmetric tensor ${}^\star\Theta^{ij}$ is

$${}^\star\Theta^{ij} = -\frac{1}{\kappa}({}^\star G^{-1}_{eff} B\, {}^\star G^{-1})^{ij}\,. \tag{25}$$

# Noncommutativity

- Poisson brackets are of the form

$$\{x^i(\sigma), x^j(\overline{\sigma})\} = {}^\star\Theta^{ij}\Delta(\sigma+\overline{\sigma})\,, \tag{26}$$

$$\{x^i(\sigma), {}^\star F(\overline{\sigma})\} = 0\,, \quad \{{}^\star F(\sigma), {}^\star F(\overline{\sigma})\} = 0\,, \tag{27}$$

where

$$\Delta(\sigma) = \begin{cases} -1 & \text{if } \sigma = 0 \\ 0 & \text{if } 0 < \sigma < 2\pi\,. \\ 1 & \text{if } \sigma = 2\pi \end{cases} \tag{28}$$

- String endpoints move along $Dp$-brane, so it is a noncommutative manifold.
- Presence of momenta in the solution for $x^i$ makes Poisson brackets to be nonzero.
- Solution for $x^i$ as well as the noncommutativity parameter depend on central charge $c$.

# Effective theory

- Using the solution and the expression for canonical Hamiltonian we obtain effective Hamiltonian

$$
\begin{aligned}
\tilde{H}_c &= \int d\sigma \tilde{\mathcal{H}}_c\,, \quad \tilde{\mathcal{H}}_c = \tilde{T}_- - \tilde{T}_+\,, \\
\tilde{T}_\pm &= \mp\frac{1}{4\kappa}\Big[({}^\star G^{-1}_{eff})^{ij}\, {}^\star\tilde{j}_{\pm i}\, {}^\star\tilde{j}_{\pm j} + \frac{\alpha}{4}\, {}^\star\tilde{j}_{\pm(F)}\, {}^\star\tilde{j}_{\pm(F)}\Big] \qquad (29)
\end{aligned}
$$

where we introduced effective currents

$$
{}^\star\tilde{j}_{\pm i} = p_i \pm \kappa\, {}^\star G^{eff}_{ij} q'^j\,, \quad {}^\star\tilde{j}_{\pm(F)} = p \pm \frac{4\kappa}{\alpha}\, {}^\star f'\,. \qquad (30)
$$

## Case (2) - $A = 0$ and $\tilde{A} \neq 0$

- For $A = 0$ metric ${}^\star G_{ij}$ is singular and its determinant has one zero.
- From the expression for canonical momenta, $\pi_i = \kappa({}^\star G_{ij}\dot{x}^j - 2B_{ij}x'^j)$, and singularity of the metric ${}^\star G_{ij}$ follows that the velocity $x_0 \equiv a_i x^i$ can not be expressed in terms of the momenta.
- Current ${}^\star j \equiv a^{i\star} j_{\pm i}$ is a primary constraint.
- Consistency procedure gives that current ${}^\star j$ is a first class constraint, and consequently, it generates gauge symmetry

$$\delta_\eta X = \{X, G\}\,, \qquad G \equiv \int d\sigma \eta(\sigma)\, {}^\star j(\sigma)\,. \tag{31}$$

- Gauge transformations

$$\begin{aligned} \delta_\eta x^i &= a^i \eta\,, \quad \delta_\eta\, {}^\star F = 0\,, \\ \delta_\eta \pi_i &= 2\kappa a^j B_{ji}\eta'\,, \quad \delta_\eta \pi = 0\,. \end{aligned} \tag{32}$$

- Good gauge condition is $x_0 \equiv a_i x^i = 0$.

# Case (3) - $\tilde{A} = 0$ and $A \neq 0$

- From Eq.(21) we have that $\det M_{AB}$ for $\tilde{A} = 0$ has two zeros.
- Singularity of matrix $M_{AB}$ is directly connected with singularity of the metric ${}^{\star}G_{ij}^{eff}$.
- Singular directions of ${}^{\star}G_{ij}^{eff}$ are $\tilde{a}^i$ and $(\tilde{a}B)^i$.
- Consequently, two constraints originating from boundary conditions turn into first class constraints

$$\Gamma_1 = \tilde{a}^i \Gamma_i \,, \quad \Gamma_2 = 2(\tilde{a}B)^i \Gamma_i \,. \tag{33}$$

- They generate local gauge symmetry and we fix the gauge

$$x_0 = 0 \,, \quad x_1 \equiv (aB)_i x^i = 0 \,. \tag{34}$$

# Solution of the cases (2) and (3)

- Solutions have common form

$$x^i_{D_p}(\sigma) = Q^i(\sigma) - 2\,{}^\star\Theta^{ij}\int_0^\sigma d\sigma_1 P_j(\sigma_1)\,, \quad \pi_i^{D_p} = P_i\,, \quad (35)$$

$$x_0\big|_0^\pi = 0\,, \quad \pi_0 = 0\,, \quad x_1\big|_0^\pi = 0\,, \quad \pi_1 = 0\,, \quad (36)$$

$${}^\star F = {}^\star f\,, \quad \pi = p\,, \quad (37)$$

where string coordinates $x^i_{Dp} = ({}^\star P_{D_p})^i{}_j x^j$ are expressed in terms of effective string variables

$$Q^i = ({}^\star P_{D_p})^i{}_j q^j\,, \quad P_i = ({}^\star P_{D_p})_i{}^j p_j\,. \quad (38)$$

# Antisymmetric tensor and projector

- Antisymmetric tensor ${}^\star\Theta^{ij}$ is given by expression

$$ {}^\star\Theta^{ij} = -\frac{1}{\kappa}(G_{eff}^{-1}\,{}^\star P_{D_p} B G^{-1\star} P_{D_p})^{ij} \,, \tag{39} $$

where

$$ ({}^\star P_{D_p})^j = \delta_i{}^j - \frac{a_i \tilde{a}^j}{\tilde{a}^2} - \frac{4}{\tilde{a}^2 - a^2}(Ba)_i(\tilde{a}B)^j \,. \tag{40} $$

projects on the subspace othogonal to the vectors $\tilde{a}^i$ and $(\tilde{a}B)^i$.

# Noncommutativity and effective theory

- Variable ${}^\star F$ decouples and it is a commutative variable, while the $Dp$-brane coordinates $X^i_{D_p}$ satisfy algebra

$$\{x^i_{D_p}(\tau,\sigma), x^j_{D_p}(\tau,\overline{\sigma})\} = {}^\star\Theta^{ij}\Delta(\sigma+\overline{\sigma})\,. \qquad (41)$$

- The number of $Dp$-brane dimensions decreases because $x_0$ and $x_1$ satisfy Dirichlet boundary conditions.
- Effective Hamiltonian has a form

$$\begin{aligned}\tilde{\mathcal{H}}_c &= \tilde{T}_- - \tilde{T}_+\,,\\ \tilde{T}_\pm &= \mp\frac{1}{4\kappa}\left[(G^{-1}_{eff}\,{}^\star P_{D_p})^{ij}\,{}^\star\tilde{j}_{\pm i}\,{}^\star\tilde{j}_{\pm j} + \frac{\alpha}{4}\,{}^\star\tilde{j}_{\pm(F)}{}^\star\tilde{j}_{\pm(F)}\right],\end{aligned}$$

where

$${}^\star\tilde{j}_{\pm i} = P_i \pm \kappa({}^\star P_{D_p} G_{eff})_{ij} Q'^j\,, \quad {}^\star\tilde{j}_{\pm(F)} = p \pm \frac{4\kappa}{\alpha}\,{}^\star f'\,. \quad (42)$$

# Conclusions

- Quantum conformal invariance is preserved even in the presence of the conformal factor of the world-sheet metric.
- For $A = 0$ metric ${}^{\star}G_{ij}$ is singular producing one standard Dirac constraint. In the case for $\tilde{A} = 0$ we have that effective metric ${}^{\star}G_{ij}^{eff}$ is singular and has two singular directions. Because the algebra of the constraints originating from boundary conditions closes on ${}^{\star}G_{ij}^{eff}$, two first class constraints appear.
- First class constraints generate local gauge symmetries which decrease the number of the $Dp$-brane dimensions.
- Canonical variables, which describe string dynamics, and noncommutativity parameter depend on the central charge $c$.
- In the limit $\alpha \to \infty \, (c \to 0)$ we obtain the results of the Liouville free case (*B. Nikolić and B. Sazdović, Phys. Rev.* **D 74** *(2006) 045024*).