# Nonarchimedean and Noncommutative Quantum Cosmology and Tachyons

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in collaboration with D. Dimitrijevic and G. Djordjevic.

- Motivation.

- Mathematics.

- Quantum Mechanics on nonarchimedean/p-adic/ spaces.

- Minisuperspace Quantum Cosmology as a Quantum Mechanics over minisuperspace.

- Kaluza-Klein (4+1) dimensional empty model.

- Tachyons

- Concluding remarks and perspectives for the future research

## MOTIVATION

- Appearance of the Nature at the small distances.

- Archimedean or nonarchimedean?

## Concept of distance.

- We measure distances (between real and rational numbers) using the absolute value on the real numbers, (natural metric on the field of the real numbers *R*).

- How did the real numbers enter analysis?

- The field of real numbers **R** is the result of *completing* the field of rationals **Q** with the respect to the usual absolute value |.|.

- The field **Q** is Causchi incomplete with respect to the usual absolute value |.|

$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, …\} => SQRT(2)$$
$$SQRT(2) \text{ notin } Q!!!$$

(A sequence $\{x_n\}$ is called a Caushy sequence with the respect to the norm |.| if it satisfies the following property: Given any *a*>0, there exists some *N* such that *m,n > N* implies $|x_m - x_n| < a$. Basically, a sequence is Caushy if its terms became "arbitrarily close" with respect to the norm |.|)

- A map |.| from the rationals to the non-negative reals is called a *norm* (or valuations) if it satisfies the three following conditions:

(1) $|x| = 0$ if and only if $x = 0$
(2) $\forall x, y$ we have $|xy| = |x||y|$ (norm of product is product of norms)
(3) $\forall x, y$ we have $|x + y| \leq |x| + |y|$ (the triangle inequality)

- The usual absolute value |.| clearly satisfies these properties, but what other kinds of norms can exist?

- There is *trivial norm* $|x|=1$ for all rationals $x$ except 0, with $|0|=0$.

- ***Besides the usual completion of field of rationals (which leads to the R) there is a non-obvious completion of rationals . These are the p-adic fields $Q_p$ where p is some fixed prime number (Kurt Hensel in 1902), (p=2,3,5,7,...).***

- Each p-adic field $\mathbf{Q}_p$ is defined by completing $\mathbf{Q}$ with respect to the "new" norm $|.|_p$ which is defined as follows:

- $0 \neq x \in Q, \quad x = p^\gamma \dfrac{m}{n}$ where $m$ and $n$ are nonzero integers, neither divisible by the prime $p$, and $\gamma$ is an integer

$$|x|_p = p^{-\gamma}.$$

- If we further define $|0|_p=0$, then that $|.|_p$ satisfies the necessary conditions above to be a norm on $\mathbf{Q}$.

- The set of equivalence classes of Caushy (with respect to $|.|_p$ ) sequences has a natural field structure. It is a completion of $\mathbf{Q}$ which we call the field of ***p-adic numbers***

**Nonarchimedean norm**

- There is a stronger inequality for an absolute value |.| than triangle inequality which is known as the **ultrametric** inequality or strong triangle inequality:

$$|x+y| \le \max\{|x|,|y|\}$$

- Any norm |.| satisfying this is called **nonarchimedean** or **ultrametric**.

- The usual norm on the real line is clearly archimedean – in fact there is an archimedean axiom:

"Any given large segment of a straight line can be surpassed by successive addition of small segments along the same line."

- A more formal statement of the axiom would be that if $0 < |x| < |y|$ then there is some positive integer $n$ such that $|nx| > |y|$.

**Can we measure distances as small as we want?**

- It is very difficult to imagine a situation where this axiom does not hold, but the archimedean axiom breaks down at the Planck scale, ($10^{-33}$ m, $10^{-44}$ s).

- Below this scale, distances and durations cannot scaled up in order to produce macroscopic distances and durations.

- In other words, we must abandon the archimedean axiom at very small distances.

- How can one construct a physical theory corresponding to a non-archimedean geometry?

geometry $\leftarrow\rightarrow$ number system

- What should be used instead of real numbers?

- In computations in everyday life, in scientific experiments and on computers we are dealing with integers and fractions, that is with rational numbers and we newer have dealings with irrational numbers.

- Results of any practical action we can express only in terms of rational numbers which are considered to have been given to us by God. What norms do exist on **Q**?

- There is a remarkable Ostrowski theorem describing all norms on **Q**. According to this theorem: *any nontrivial norm on **Q** is equivalent to either ordinary absolute value or p-adic norm for some fixed prime number p.*

- *p*-adic fields are **nonarchimedean**, it is natural to consider formulations of physics in terms of $\mathbf{Q}_p$ rather than **R**

- "Which *p* do we use?"

- The adeles constitute a locally compact topological ring $\mathbf{A_Q}$, individually taking the form

$$(a_\infty; a_2, a_3, a_5, \cdots)$$

In all but a finite number of cases $a_p$ is a p-adic integer ($|a_p|_p \leq 1$).

## *p*-ADIC FUNCTIONS AND INTEGRATION

- There are primary two kinds of analyses on $\mathbf{Q}_p$ : $\mathbf{Q}_p \rightarrow \mathbf{Q}_p$ (class.) and $\mathbf{Q}_p \rightarrow \mathbf{C}$ (quant.).

- Usual complex valued functions of p-adic variable, which are employed in mathematical physics, are :

(i)     an additive character $\chi_p(x) = \exp 2\pi i \{x\}_p$, where $\{x\}_p$ is the fractional part of $x \in Q_p$,

(ii)     multiplicative character $\pi_s(x) = |x|_p^s$, where $s \in C$, and

(iii)     locally constant functions with compact support, like $\Omega(|x|_p)$, where

$$\Omega(|x|_p) = \begin{cases} 1 & |x|_p \leq 1 \\ 0 & |x|_p > 1 \end{cases}.$$

- There is well defined Haar measure and integration. Important integrals are

$$\int_{Q_p} \chi_p(ayx)dx = \delta_p(ay) = |a|_p^{-1} \delta_p(y), a \neq 0$$

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x)dx = \lambda_p(\alpha)|2\alpha|_p^{-1/2} \chi_p\left(-\frac{\beta^2}{4\alpha}\right), \alpha \neq 0,$$

- The number theoretic function $\lambda_p(x)$ is a map $\lambda_p : Q_p^* \rightarrow C$ defined as follows:

$$\lambda_p(x) = \begin{cases} 1, & m = 2j & p \neq 2 \\ \left(\frac{x_0}{p}\right), & m = 2j+1 & p \equiv 1 (\mathrm{mod}\, 4) \\ i\left(\frac{x_0}{p}\right), & m = 2j+1 & p \equiv 3 (\mathrm{mod}\, 4) \end{cases},$$

$$\lambda_2(x) = \begin{cases} \frac{1}{\sqrt{2}}[1 + (-1)^{x_1} i] & m = 2j \\ \frac{1}{\sqrt{2}}(-1)^{x_1 + x_2}[1 + (-1)^{x_1} i] & m = 2j + 1 \end{cases},$$

$x = p^m(x_0 + x_1 p + x_2 p^2 + \cdots)$, and $m, j \in Z$.

- $(\frac{x_0}{p})$ is the Legandre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & if \quad a \equiv y^2 (\bmod\, p) \\ -1 & if \quad a \neq y^2 (\bmod\, p) \\ 0 & if \quad a \equiv 0 (\bmod\, p) \end{cases}.$$

Properties of $\lambda_p$-function

$$\lambda_p(a^2 x) = \lambda_p(x), \quad \lambda_p(x)\lambda_p(-x) = 1, \quad \lambda_p(x)\lambda_p(y) = \lambda_p(x+y)\lambda_p(x^{-1} + y^{-1})$$

$$|\lambda_p(x)|_\infty = 1, \quad a \neq 0.$$

Real analogues of integrals

$$\int_{Q_\infty} \chi_\infty(ayx)dx = \delta_\infty(ay) = |a|_\infty^{-1} \delta_\infty(y), a \neq 0$$

$$\int_{Q_\infty} \chi_\infty(\alpha x^2 + \beta x)dx = \lambda_\infty(\alpha) | 2\alpha |_\infty^{-1/2} \chi_\infty\left(-\frac{\beta^2}{4\alpha}\right), \alpha \neq 0,$$

$$Q_\infty \equiv R, \quad \chi_\infty(x) = \exp(-2\pi i x)$$

$$\lambda_\infty(x) = \sqrt{\frac{sign\, x}{i}}$$

## QUANTUM MECHANICS ON ADELIC SPACES

Reasons to use p-adic numbers and adeles in quantum physics:

- the field of rational numbers **Q**, which contains all observational and experimental numerical data, is a dense subfield not only in **R** but also in the fields of p-adic numbers $\mathbf{Q}_p$

- there is an analysis within and over $\mathbf{Q}_p$ like that one related to **R**

- general mathematical methods and fundamental physical laws should be invariant [I.V. Volovich, *Number theory as the ultimate physical theory,* CERN preprint, CERN-TH.4781/87 (July 1987)] under an interchange of the number fields **R** and $\mathbf{Q}_p$

- there is a quantum gravity uncertainty ($\Delta x \geq l_0 = \sqrt{\hbar G / c^3}$), when measures distances around the Planck length $l_0$, which restricts priority of Archimedean geometry based on the real numbers and gives rise to employment of non-Archimedean geometry related to p-adic numbers

- it seems to be quite reasonable to extend standard Feynman's path integral method to non-Archimedean spaces,

- adelic quantum mechanics [B. Dragovich, Adelic Model of Harmonic Oscillator, *Theor,. Math. Phys,* **101** (1994) 1404-1412] is consistent with all.

Adelic quantum mechanics

$$(L_2(A), W(z), U(t))$$

- ♦ $L_2(A)$ - adelic Hilbert space,

- ♦ $W(z)$ - Weyl quantization of complex-valued functions on adelic classical phase space,

- ♦ $U(t)$ - unitary representation of an adelic evolution operator on $L_2(A)$

## Dynamics of p-adic quantum model?

$$U_p(t)\psi^{(p)}(x) = \int_{Q_p} K_t^{(p)}(x,y)\psi^{(p)}(y)dy$$

*p*-adic quantum mechanics is given by a triple

$$(L_2(Q_p), W_p(z_p), U_p(t_p))$$

Adelic evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_A K_t(x,y)\psi(y)dy = \prod_{v=\infty,2,3,...,p,...} \int_{Q_v} K_t^v(x_v,y_v)\psi^{(v)}(y_v)dy_v$$

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_\alpha(x) = \chi(E_\alpha t)\psi_\alpha(x)$$

The main problem in our approach is computation of p-adic propagator $K_p(x'',t'';x',t')$ in Feynman's path integral method, i.e.

$$K_p(x'',t'';x',t') = \int_{(x',t')}^{(x'',t'')} \chi_p(-\tfrac{1}{h}\int_{t'}^{t''} L(\dot{q},q,t))Dq$$

Exact general expression for propagator

$$K_p(x'',t'';x',t') = \lambda_p(-\tfrac{1}{2h}\tfrac{\partial^2 \bar{S}}{\partial x'\partial x''}) \times \left|\tfrac{1}{h}\tfrac{\partial^2 \bar{S}}{\partial x'\partial x''}\right|_p^{1/2} \chi_p\left(-\bar{S}(x'',t'';x',t')\right)$$

$$K_p(x'',y'',z'',t'';x',y',z',t') = \lambda_p\left(\det\begin{pmatrix} -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial x'\partial x''} & -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial x'\partial y''} & -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial x'\partial z''} \\ -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial y'\partial x''} & -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial y'\partial y''} & -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial y'\partial z''} \\ -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial z'\partial x''} & -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial z'\partial y''} & -\tfrac{1}{2}\tfrac{\partial^2 \bar{S}}{\partial z'\partial z''} \end{pmatrix}\right)$$

$$\times \left|\det\begin{pmatrix} -\tfrac{\partial^2 \bar{S}}{\partial x'\partial x''} & -\tfrac{\partial^2 \bar{S}}{\partial x'\partial y''} & -\tfrac{\partial^2 \bar{S}}{\partial x'\partial z''} \\ -\tfrac{\partial^2 \bar{S}}{\partial y'\partial x''} & -\tfrac{\partial^2 \bar{S}}{\partial y'\partial y''} & -\tfrac{\partial^2 \bar{S}}{\partial y'\partial z''} \\ -\tfrac{\partial^2 \bar{S}}{\partial z'\partial x''} & -\tfrac{\partial^2 \bar{S}}{\partial z'\partial y''} & -\tfrac{\partial^2 \bar{S}}{\partial z'\partial z''} \end{pmatrix}\right|_p^{1/2} \chi_p\left(-\bar{S}(x'',y'',z'',t'';x',y',z't')\right)$$

where $\bar{S}(x'',y'',z'',t'';x',y',z't')$ is the action for classical trajectory.

## Illustration of exactly soluble p-adic and adelic quantum mechanical models,

- a free particle and harmonic oscillator `[VVZ, Dragovich]`

- a particle in a constant field, `[Djordjevic, Dragovich]`

- a free relativistic particle`[Djordjevic, Dragovich, Nesic]`

- a harmonic oscillator with time-dependent frequency `[Djordjevic, Dragovich]`

**Resume of AQM:** AQM takes in account ordinary as well as p-adic effects and may me regarded as a starting point for construction of more complete quantum cosmology and string/M theory. In the low energy limit AQM effectively becomes the ordinary one.

## The form of adelic wave function

$$\psi = \psi_{\infty}(x_{\infty}) \cdot \prod_{p \in M} \psi_p(x_p) \cdot \prod_{p \notin M} \Omega(|x|_p)$$

## `ADELIC QUANTUM COSMOLOGY

-   The main task of AQC is to describe the very early stage in the evolution of the Universe.

-   At this stage, the Universe was in a quantum state, which should be described by a wave function (complex valued and depends on some real parameters).

-   But, QC is related to Planck scale phenomena - it is natural to reconsider its foundations.

-   We maintain here the standard point of view that the wave function takes complex values, but we treat its arguments in a more complete way!

-   We regard space-time coordinates, gravitational and matter fields to be adelic, i.e. they have real as well as p-adic properties simultaneously.

-   There is no Schroedinger and Wheeler-De Witt equation for cosmological models.

-   **Feynman's path integral method was exploited [Drag. Nes.] and minisuperspace cosmological models are investigated as a model of adelic quantum mechanics [Drag. Djordj. Nes. Vol.].**

-   Adelic minisuperspace quantum cosmology is an application of adelic quantum mechanics to the cosmological models.

-   Path integral approach to standard quantum cosmology, the starting point is Feynman's path integral method

$$< h_{ij}^{''}, \phi^{''}, \Sigma^{''} \mid h_{ij}^{'}, \phi^{'}, \Sigma^{'} >_\infty = \int D(g_{\mu\nu})_\infty D(\phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu}, \phi])$$

p-adic complex valued cosmological amplitude

$$< h_{ij}^{''}, \phi^{''}, \Sigma^{''} \mid h_{ij}^{'}, \phi^{'}, \Sigma^{'} >_p = \int D(g_{\mu\nu})_p D(\phi)_p \chi_p(-S_p[g_{\mu\nu}, \phi])$$

## Kaluza-Klein (4+1) - dimensional "empty" model

- FRW metrics (F. Darabi, *Phys.Lett.* **B615** (2005) 141-145)

$$ds^2 = -N^2 dt^2 + R^2(t) \frac{dr_i dr^i}{\left(1 + \frac{kr^2}{4}\right)^2} + a^2(t) d\rho^2$$

- Einstein – Hilbert action for empty model

$$S = \int \sqrt{-g} \, (\tilde{R} - \Lambda) \, dt \, d^3r \, d\rho$$

- Lagrangian

$$L = \frac{1}{2N} Ra\dot{R}^2 + \frac{1}{2N} R^2 \dot{R}\dot{a} - \frac{1}{2} kNRa + \frac{1}{6} N\Lambda R^3 a$$

## Commutative model over real space

- By defining $\omega^2 = -\frac{2\Lambda}{3}$ $(\Lambda < 0)$

- Changing $u = \frac{1}{\sqrt{8}}[R^2 + Ra - \frac{3k}{\Lambda}]$, $v = \frac{1}{\sqrt{8}}[R^2 - Ra - \frac{3k}{\Lambda}]$

- Lagrangian (oscillator-ghost-oscillator system)

$$L = \frac{1}{2N}(\dot{u}^2 - N^2\omega^2 u^2) - \frac{1}{2N}(\dot{v}^2 - N^2\omega^2 v^2)$$

- Classical action

$$\bar{S}(u'',v'',N;u',v',0) = \frac{1}{2}\omega[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u''\frac{2}{\sin N\omega})]$$

$$K(u'',v'',N;u',v',0) = \frac{\omega}{\sin N\omega}\exp[2\pi i\bar{S}(u'',v'',N;u',v',0)]$$

- Wheeler-DeWitt equation

$$\left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - \omega^2 u^2 + \omega^2 v^2 \right) \Psi(u,v) = 0$$

- Oscillator-ghost-oscillator solutions belonging to the Hilbert space $H^{(m_1,m_2)}(L^2)$ as

$$\Psi^{(m_1,m_2)}(u,v) = \sum_{l=0}^{\infty} c_l \Phi_l^{(m_1,m_2)}(u,v)$$

with $m_1, m_2 \geq 0$ and $c_l \in C$. The basis solutions $\Phi_l^{(m_1,m_2)}(u,v)$ are separable as

$$\Phi_l^{(m_1,m_2)}(u,v) = \alpha_{m_2+(2m_2+1)l}()u\beta_{m_1+(2m_1+1)l}(v)$$

with normalized solutions

$$\alpha_n(u) = \left( \frac{\omega}{\pi} \right)^{1/4} \frac{e^{-\omega u^2/2}}{\sqrt{2^n n!}} H_n(u\sqrt{\omega})$$

$$\beta_n(v) = \left( \frac{\omega}{\pi} \right)^{1/4} \frac{e^{-\omega v^2/2}}{\sqrt{2^n n!}} H_n(v\sqrt{\omega})$$

## Commutative Kaluza-Klein (4+1) - dimensional "empty" model over p-adic space

$$ds^2 = -N^2 dt^2 + R^2(t)\frac{dr_i dr^i}{\left(1+\frac{kr^2}{4}\right)^2} + a^2(t)d\rho^2$$

$$S = \int \sqrt{-g}\,(\tilde{R} - \Lambda)\,dt\,d^3r\,d\rho$$

- P-adic Lagrangian in the minisuperspace $(R, a)$

$$L = \frac{1}{2N}Ra\dot{R}^2 + \frac{1}{2N}R^2\dot{R}\dot{a} - \frac{1}{2}kNRa + \frac{1}{6}N\Lambda R^3 a$$

- P-adic Classical action

$$\overline{S}_p(u'', v'', N; u', v', 0)$$

$$= \frac{1}{2}\omega[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u''\frac{2}{\sin N\omega})]$$

- The propagator is

$$K_p(u'', v'', N; u', v', 0)$$

$$= \frac{1}{|N|_p}\chi_p\left(\frac{\omega(u''^2 + u'^2 - v'^2 - v''^2)}{2\tan N\omega} + \frac{\omega(v'v'' - u'u'')}{\sin N\omega}\right)$$

- In the region of convergence $G_p = \{x \in Q_p : |x|_p \le |2p|_p\}$

- $\Psi_p(u, v) = \Omega(|u|_p)\Omega(|v|_p)$

- $\Psi_p(u, v) = \Omega(p^\nu |u|_p)\Omega(p^\mu |v|_p), \nu, \mu = 1,2,3,...$

$$\Psi_p(u, v) = \begin{cases} \delta(p^\nu - |u|_p)\delta(p^\mu - |v|_p) & |N|_p \le p^{2\nu-2}, |N|_p \le p^{2\mu-2} \\ \delta(2^\nu - |u|_2)\delta(2^\mu - |v|_2) & |N|_2 \le 2^{2\nu-3}, |N|_2 \le 2^{2\mu-3} \end{cases},$$

## Discretization

It is shown that there is possibility of construction adequate p-adic (4+1)-dim empty Kaluza-Klein model.

It is shown that there exist adelic ground states of the form

$$\psi(u,v) = \psi_\infty^\pm(u_\infty, v_\infty) \prod_{p \in M} \psi_p(u_p, v_p) \prod_{p \notin M} \Omega(|u_p|_p)\Omega(|v_p|_p)$$

- Adopting the usual probability interpretation of the wave function, we have

$$|\psi(u,v)|_\infty^2 = |\psi_\infty^\pm(u_\infty, v_\infty)|_\infty^2 \prod_{p \in M} |\psi_p(u_p, v_p)|_\infty^2 \prod_{p \notin M} \Omega(|u_p|_p)\Omega(|v_p|_p)$$

$$(\Omega(|u|_p))^2 = \Omega(|u|_p)$$

- At the rational points $u$ and $v$ and for $M = \varnothing$

$$|\psi(u,v)|_\infty^2 = \begin{cases} |\psi_\infty^\pm(u,v)|_\infty^2 & u,v \in Z \\ 0 & u,v \in Q \setminus Z \end{cases}$$

- Discretization of minisuperspace coordinates

## Noncommutative case

- In the previous section we assume that in $(u, v)$ minisuperspace holds

$$[u, v] = 0, \quad [u, p_u] = [v, p_v] = i\hbar, \quad [p_u, p_v] = 0,$$

$$p_u = \frac{\dot{u}}{N}, \ p_v = \frac{\dot{v}}{N}$$

- Noncommutative case - the same Lagrangian, a new algebra

$$[u, v] = i\theta, \quad [u, p_u] = [v, p_v] = i\hbar, \quad [p_u, p_v] = 0.$$

- Transformation

$$u = u - \frac{\theta}{2} p_v, \, v = v + \frac{\theta}{2} p_u$$

- Model as commuting one with Lagrangian

$$L_\theta = \frac{\omega^2}{\omega_\theta^2} \left[ \frac{1}{2N} (\dot{u}^2 - N^2 \omega_\theta^2 u^2) - \frac{1}{2N} (\dot{v}^2 - N^2 \omega_\theta^2 v^2) + \frac{1}{2N} \omega_\theta^2 \theta (\dot{u}v + u\dot{v}) \right]$$

$$\omega_\theta^2 = \frac{\omega^2}{1 + \frac{\omega^2 \theta^2}{4}}$$

- Equations of motion

$$\ddot{u} + N^2 \omega_\theta^2 u = 0, \quad \ddot{v} + N^2 \omega_\theta^2 v = 0$$

- Solutions

$$u(t) = A \cos N\omega_\theta t + B \sin N\omega_\theta t, \quad v(t) = C \cos N\omega_\theta t + D \sin N\omega_\theta t$$

- Classical action

$$\bar{S}_\theta(u'',v'',N;u',v',0) = \frac{1}{2}\omega\sqrt{1+\frac{\omega^2\theta^2}{4}}$$

$$\left[(u''^2+u'^2-v''^2-v'^2)\cot N\omega_\theta - (u'u''-v'v'')\frac{2}{\sin N\omega_\theta} + \frac{\theta\omega_\theta}{N}(u''v''-u'v')\right]$$

- Quantum propagator

$$K_\theta(u'',v'',N;u',v',0) = \sqrt{1+\frac{\omega^2\theta^2}{4}}\sqrt{\frac{\omega^2}{\sin^2 N\omega_\theta}}\exp(2\pi i\bar{S}_\theta(u'',v'',N;u',v',0))$$

- Commutative regime - $\theta = 0$

$$K_0(u'',v'',N;u',v',0) = \frac{\omega}{\sin N\omega}\chi_\infty(-\bar{S}(u'',v'',N;u',v',0))$$

## TACHYONS

- *mass$^2$* < 0.

- Such particles are called tachyons. For such a theory vacuum state is generally unstable.

- A. Sen proposed a field theory of tachyon matter [A. Sen, *JHEP* **0204**, 048, 2002., A. Sen, *Tachyon Dynamics in Open String Theory*, hep-th/0410103].

$$S = -\int d^{n+1}x V(T)\sqrt{1 + \eta^{ij}\partial_i T \partial_j T}$$

- $\eta_{00} = -1$, $\eta_{\mu\nu} = \delta_{\mu\nu}$, $\mu,\nu = 1,...,n$,

- $T(x)$ - scalar tachyon field,

- $V(T)$ is the tachyon potential

$$V(T) = e^{-\alpha T(x)}.$$

- Lower dimensional analogs of this tachyon field theory [S. Kar, *A simple mechanical analog of the field theory of tachyon matter*, hep-th/0210108.].

**Zero dimensional case**

$$x^i \rightarrow t, \ T \rightarrow x, \ V(T) \rightarrow V(x).$$

- The action:

$$S = -\int dt V(x)\sqrt{1 - \dot{x}^2},$$

- Equation of motion, including $V(x) = e^{-\alpha x}$

$$\ddot{x} + \alpha\dot{x}^2 = \alpha$$

- This equation coincide with the equation for the system under gravity in the presence of quadratic damping:

$$m\ddot{y} + \beta\dot{y}^2 = mg \qquad (*)$$

- This equation can be derived from the action:

$$S = -\int dt e^{-\frac{\beta y}{m}}\sqrt{1 - \frac{\beta}{mg}\dot{y}^2} \ . \ \text{a)}$$

- The solution can be found perturbatively:

$$y(t) = y_0(t) + y_1(t),$$

$$m\ddot{y}_0 = mg$$

($y_0(t)$ is solution of Eq. (*) for $\beta = 0$)

- $y_1(t)$ is obtained from the same equation after inserting $y_0(t)$ and neglecting all non linear terms:

$$\ddot{y}_1 + at\dot{y}_1 = -bt^2, \quad \left(a = \frac{2\beta g}{m}, b = \frac{\beta g^2}{m}\right).$$

- For $y_0(0) = 0$ and $\dot{y}_0(0) = 0$ **the final solution** is given by:

$$y(t) = \frac{g}{2}t^2 + \frac{b}{2a}t^2({}_2F_2[1,1;\frac{3}{2},2;-\frac{at^2}{2}]-1),$$

$${}_2F_2[1,1;\frac{3}{2},2;-\frac{at^2}{2}] \text{ is hypergeometric function.}$$

- For small $t$

$$y(t) = \frac{g}{2}t^2 - \frac{\beta g^2}{12m}t^4 \ .$$

**Feynman path integral**

- The kernel $K(y'',T;y',0)$ of the unitary evolution operator $U(0,T)$,

$$K(y'',T;y',0) = \int Dy\, e^{\frac{2\pi i}{h}\int_0^T L dt}.$$

- Semi-classical expression for the kernel if the classical action $\overline{S}(y'',T;y',0)$ is polynomial quadratic in $y'$ and $y''$ (which holds for both real and p-adic number fields

$$K(y'',T;y',0) = \left(\frac{i}{h}\frac{\partial^2 \overline{S}}{\partial y' \partial y''}\right)^{1/2} e^{\frac{2\pi i}{h}\overline{S}(y'',T;y',0)}.$$

- For small $\beta$

$$S \xrightarrow[\beta \to 0]{} S' = -\int dt[\frac{\beta}{2mg}\dot{y}^2 + \frac{\beta}{m}y - 1],$$

This action is quadratic with respect to velocity, and standard procedure can be engaged for the path integral.

## CONCLUSION

- Sen`s proposal and similar conjectures have attracted important interests among physicists.

- Our understanding of tachyon matter, especially its quantum aspects is still quite pure.

- Perturbative solutions for classical particles analogous to the tachyons offer many possibilities for further investigations and toy models in quantum mechanics, quantum and string field theory and cosmology on archimedean and nonarchimedean spaces.