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Solitons and giants in matrix model(s)

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- I. Andrić, L. J., D. Jurman, Solitons and excitations in the duality-based matrix model, JHEP 0508 (2005) 064, hep-th/0411179
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Plan of the talk

- Introduction
- Matrix model and the collective-field formulation
  - Conformal invariance
- Matrix model, Riccati equation and boundary fields
- Semiclassical solutions
- Quantum excitations around semiclassical solutions
- Discussion and interpretation
- Duality-based matrix model

Introduction

- AdS/CFT correspondence
- The interpretation of the matrix eigenvalues as fermions allows a description of gravitational excitations in the holographic dual of N = 4 SYM in terms of droplets in the phase space occupied by fermions. The giant gravitons expanding along $\boldsymbol{AdS_5}$ and $\boldsymbol{S^5}$ could be interpreted as a single excitation high above the Fermi sea, or as a hole in the Fermi sea, respectively (Berenstein)
- The correspondence between the general fermionic droplet and the classical ansatz for the AdS configuration (LLM)
- The matrix model with the harmonic-oscillator potential is related to the free matrix model via $\boldsymbol{su(1,1)}$ algebra which contains Hamitonians of both models as generators. As a consequence, their eigenstates are related via coherent states or by time reparametrization.

Matrix model and the collective-field formulation

The dynamics of the one-matrix model is defined by the action (BIPZ)

$$S = \int dt \left(\frac{1}{2} Tr \dot{M}^2(t) - V(M) \right)$$

with the $N \times N$ matrix M being

M=R real symmetric O(N) invariant
M=H complex hermitian U(N) invariant
M=Q quaternionic real Sp(N) invariant

$$\begin{aligned} M &= U \Lambda U^\dagger \\ \dot{M} &= U \left(\dot{\Lambda} + [U^{-1}\dot{U}, \Lambda] \right) U^\dagger \end{aligned}$$

Conserved quantity:

$$J = [M, \dot{M}] = U \left([\Lambda, \dot{\Lambda}] + [\Lambda, [U^{-1}\dot{U}, \Lambda]] \right) U^\dagger$$

$$Tr \dot{M}^2 = Tr \dot{\Lambda}^2 + \sum_{a \neq b} \frac{(U^{-1} J U)_{ab} (U^\dagger J U)_{ba}}{(\lambda_a - \lambda_b)^2}$$

$$M = \begin{pmatrix} \lambda_1 & \dots & \\ & & \\ & & \lambda_N \end{pmatrix} \quad \rightarrow \quad x_1 \dots x_N$$

On a singlet subspace the free matrix model reduces to the quantum mechanics of the N eigenvalues of the matrix M. The dynamics of the eigenvalues is determined by the QM Hamiltonian

$$H_{\rm QM} = -\frac{1}{2} \sum_i \frac{d^2}{dx_i^2} + \lambda(\lambda - 1) \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

Introduction of the invariant measure over the matrix configuration space into the wavefunctions produces a prefactor $\prod_{i<j}^N (x_i - x_j)^\lambda$. λ determines the

number of independent matrix elements n_λ in the case of real-symmetric, hermitian and quaternionic-real matrices:

$$n_\lambda = \lambda N(N-1) + N$$

and $\lambda = 1/2,\ 1,\ 2,$ respectively.
In the large-N limit, we introduce the collective field variables (Jevicki, Sakita)

$$\rho_k(t) = Tr e^{-ikM(t)}, \quad \rho(x,t) = \int \frac{dk}{2\pi} e^{ikx} \rho_k(t)$$

The free matrix Hamiltonian on singlet space

$$H = \frac{1}{2} \int \int dx dy \pi(x) \Omega[\rho; x, y] \pi(y) - \frac{i}{2} \int dx \omega[\rho; x] \pi(x)$$

where

$$\int dx \rho(x) = N, \quad \pi(x) = -i\delta/\delta\rho(x)$$

and Ω and ω are to be determined by transformation from quantum mechanics to collective field theory. Using the chain rule

$$\frac{\partial}{\partial m_\alpha^{ij}} \rightarrow \int dx \frac{\partial \rho(x)}{\partial m_\alpha^{ij}} \frac{\delta}{\delta \rho(x)}$$

one finds

$$\Omega[\rho; x, y] = \partial_{xy}^2 \left[\delta(x-y)\rho(y)\right]$$
$$\omega[\rho; x] = (\lambda - 1)\partial_x^2 \rho(x) + 2\lambda \partial_x \rho(x) ⨍ dy \frac{\rho(y)}{x-y}$$

After hermitization

$$H = \frac{1}{2} \int dx \rho(x) \left(\frac{\lambda - 1}{2} \frac{\partial_x \rho(x)}{\rho} + \lambda ⨍ dy \frac{\rho(y)}{x-y} \right)^2 - \mu \int dx \rho(x)$$
$$+ \frac{1}{2} \int dx \rho(x) (\partial_x \pi)^2 - \frac{\lambda - 1}{4} \int dx \partial_x^2 \delta(x-y)|_{y=x} - \frac{\lambda}{2} ⨍ dx \partial_x \frac{1}{x-y}|_{y=x}$$

Conformal invariance

Corresponding Lagrangian density

$$\mathcal{L}(\rho,\dot\rho)=\frac{1}{2}\frac{(\partial_x^{-1}\dot\rho)^2}{\rho}-\frac{1}{2}\rho\left[\frac{(\lambda-1)}{2}\frac{\partial_x\rho}{\rho}+\lambda \fint dy\frac{\rho(y)}{x-y}\right]^2$$

where $\partial_x^{-1}\dot\rho$ is short for $\int^x dy\dot\rho(y)$.

The action possesses three kinds of symmetry: time translation, scaling and special conformal tranformation:

$$t'=t-\epsilon t^n$$

for $n=0,1,2$, respectively. Under these transformations

$$x'=\left(\frac{\partial t'}{\partial t}\right)^{d_x}x$$
$$\rho'(x',t')=\left(\frac{\partial t'}{\partial t}\right)^{d_\rho}\rho(x,t)$$
$$dx'dt'=\left(\frac{\partial t'}{\partial t}\right)^{d_x+1}dxdt$$

and

$$d_x=1/2,\quad d_\rho=-1/2$$

The conserved charges

$$Q=\int dx\frac{\delta\mathcal{L}}{\delta\dot\rho}\delta\rho-A(\rho,\dot\rho),\;;\;\delta S=\int dt\frac{dA(\rho,\dot\rho)}{dt}$$

One calculates

$$\delta\rho=\rho'(x,t)-\rho(x,t)=(-d_\rho nt^{n-1}+d_x nt^{n-1}x\partial_x+t^n\partial_t)\rho(x,t)$$

and

$$A=-\frac{n(n-1)}{4}\int dx x^2\rho+\frac{t^n}{2}\int dx\mathcal{L}$$

For $n = 0, 1, 2$

$$
\begin{aligned}
Q_0 &= H \equiv Q_T \\
Q_1 &= -\frac{1}{2}\int dx \rho(x)\partial_x \pi(x) + tH \equiv Q_S \\
Q_2 &= \frac{1}{2}\int dx x^2 \rho(x) - t\int dx x \rho(x)\partial_x \pi(x) + \frac{t^2}{2}H \equiv Q_C
\end{aligned}
$$

These conserved quantities close the algebra of the conformal group in one dimension with respect to the classical Poisson brackets

$$\{Q_T, Q_S\}_{PB} = Q_T \ \ , \{Q_C, Q_S\}_{PB} = -Q_C \ \ , \{Q_T, Q_C\}_{PB} = 2Q_T$$

$$
\begin{aligned}
Q_0(t=0) &\rightarrow T_+ \\
Q_1(t=0) &\rightarrow T_0 \qquad SU(1,1): \ [T_+, T_-] = -2T_0 \\
Q_2(t=0) &\rightarrow T_- \qquad\qquad\qquad\quad [T_0, T_\pm] = \pm T_\pm
\end{aligned}
$$

From zero-energy eigenfunctional construct eigenfunctional of energy E as a coherent state of Barut-Girardello type, using spectrum generating $SU(1,1)$ algebra.

Matrix model, Riccati equation and boundary fields

The leading part of the collective-field Hamiltonian in the $1/N$ expansion

$$V_{eff} = \frac{1}{2}\int dx\rho(x)\left[\frac{\lambda-1}{2}\frac{\partial_x\rho(x)}{\rho(x)} - \lambda\pi\rho^H(x)\right]^2$$

where

$$\rho^H(x) = -\frac{1}{\pi}\fint dy\frac{\rho(y)}{x-y}$$

The effective potential can be rewritten as

$$\begin{aligned} V_{eff} \ &= \ \frac{1}{2}\int dx\rho(x)\left[\frac{\lambda-1}{2}\frac{\partial_x\rho(x)}{\rho(x)} + \frac{q(1-\lambda)}{2}\mathcal{P}\cot\left(\frac{qx}{2}+\varphi\right) - \right. \\ &- \ \left.\lambda\pi\rho^H(x)\right]^2 + E_0, \end{aligned}$$

where

$$\mathcal{P}\cot(qx/2+\varphi) = \lim_{\epsilon\to 0}\frac{\sin(qx+2\varphi)}{\cosh\epsilon - \cos(qx+2\varphi)}$$

Assuming the compact support $[-L/2, L/2]$, using $\int dx\rho(x) = N$ and the identity

$$(f^H g + f g^H)^H = f^H g^H - fg + f_0 g_0, \ \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = \frac{1}{L}\int dx\begin{pmatrix} f(x) \\ g(x)\end{pmatrix}$$

one obtains

$$\begin{aligned} E_0 = \quad & \frac{qN(1-\lambda)}{8}\left[(1-\lambda)q + 2\pi\lambda\frac{N}{L}\right] + \\ + \quad & \frac{q(\lambda-1)^2}{4}\rho(x)\mathcal{P}\cot\left(\frac{qx}{2}+\varphi\right)\Big|_{-L/2}^{L/2} + \\ + \quad & \frac{q\pi\lambda(\lambda-1)L}{4}\left(\rho^2(x) - {\rho^H}^2(x)\right)\Big|_{x=-2\varphi/q} \end{aligned}$$

q and φ are free parameters to be determined by boundary conditions such that the last two terms should vanish and by the condition that E_0 should be a non-negative constant. The contribution of V_{eff} to the Hamiltonian is minimized by a solution of

$$\partial_x \rho = q\mathcal{P}\cot\left(\frac{qx}{2}+\varphi\right)\rho + \frac{\lambda\pi}{\lambda-1}2\rho\rho^H$$

Find the equation for ρ^H,

$$\partial_x \rho^H = q\mathcal{P}\cot\left(\frac{qx}{2}+\varphi\right)\rho^H - q\rho_0 - \frac{\lambda\pi}{\lambda-1}\left(\rho^2 - {\rho^H}^2 - \rho_0^2\right)$$

Construct the field Φ containing only the positive frequency part of ρ

$$\Phi = \rho^H + i\rho = \frac{1}{\pi}\int dz \frac{\rho(z)}{z-x-i\epsilon}$$

and satisfying the Riccati differential equation

$$\partial_x \Phi = \frac{\lambda\pi}{\lambda-1}\Phi^2 + q\mathcal{P}\cot\left(\frac{qx}{2}+\varphi\right)\Phi + \frac{\lambda\pi\rho_0^2}{\lambda-1} - q\rho_0$$

If Φ satisfies

$$\Phi^H(x) = i\Phi(x) + \rho_0$$

then

$$\rho = -i(\Phi - \Phi^*)/2$$

Semiclassical solutions

1) The case $\lambda < 1$

$$\Phi_s(x) = \frac{iq(1-\lambda)}{\lambda\pi(e^t-1)}\frac{1-e^{i(qx+2\varphi)}}{1-e^{-t}e^{i(qx+2\varphi)}}$$
$$\rho_s(x) = \frac{q(1-\lambda)\coth(t/2)}{2\pi\lambda}\frac{1-\cos(qx+2\varphi)}{\cosh t-\cos(qx+2\varphi)}$$
$$E_0 = \frac{(1-\lambda)\pi^2}{2L^2}[\lambda N^2 M + (1-\lambda)NM^2]$$

where

$$e^t = 1 + \frac{q(1-\lambda)}{\lambda\pi\rho_0}$$

With the boundary conditions

$$\rho^H(-\frac{2\varphi}{q}) = \rho(-\frac{2\varphi}{q}) = 0, \ \mathcal{P}\cot(\frac{qL}{4}+\varphi) = 0$$

we find

$$q = 2\pi M/L, \ M \in \mathbb{N}$$

where the number M can be interpreted as the number of solitons. In order to have odd M, we take $\varphi = 0$, whereas for even M we take $\varphi = \pi/2$. Taking into account the normalization condition, we find

$$e^t = 1 + \frac{2M(1-\lambda)}{N\lambda}$$

From the M-soliton solution in the limit $L \longrightarrow \infty$, keeping ρ_0 fixed and defining

$$b = (1-\lambda)/(\lambda\pi\rho_0)$$

we find the one-soliton solution ($M = 1$, $\varphi = 0$).

$$\Phi_s(x) = \frac{1-\lambda}{\lambda\pi b}\frac{ix}{x+ib}$$
$$\rho_s(x) = \frac{1-\lambda}{\lambda\pi b}\frac{x^2}{x^2+b^2}$$
$$E_0 = \frac{(1-\lambda)^3}{2\lambda b^2}$$

The uniform zero-energy solution $\rho(x) = \rho_0$ is obtained in the limit $q \longrightarrow 0$, taking $\varphi = \pi/2$.

2) The case $\lambda > 1$
We take $q = 0$, $\varphi = \pi/2$, thus eliminating the term $\mathcal{P}\cot$ from V_{eff}, and obtain a general solution

$$\Phi_s(x) = \frac{ik(\lambda-1)}{2\pi\lambda}\frac{1+e^{-t}e^{ikx}}{1-e^{-t}e^{ikx}}$$
$$\rho_s(x) = \frac{k(\lambda-1)}{2\pi\lambda}\frac{\sinh t}{\cosh t - \cos kx}$$
$$E_0 = 0$$

with $k = 2\pi M/L$ and non-negative free parameter t.

In limit $L \longrightarrow \infty$, taking $t = 2\pi b/L$ we obtain

$$\Phi_s(x) = \frac{1-\lambda}{\lambda\pi}\frac{1}{x+ib}$$
$$\rho_s(x) = \frac{\lambda-1}{\lambda\pi}\frac{b}{x^2+b^2}$$
$$E_0 = 0$$

In the case $t \longrightarrow \infty$, we obtain the constant density solution $\rho(x) = \rho_0$.

Taking into account the normalization condition we obtain that the number of solitons M exceeds the number of particles N giving us the relation

$$\lambda = M/(M-N)$$

Quantum excitations around semiclassical solutions

Expand the Hamiltonian around the semiclassical solution

$$\rho(x,t) = \rho_s(x) + \partial_x \eta(x,t)$$

up to the quadratic terms in η

$$H^{(2)} = \frac{1}{2}\int dx \rho_s(x) A^\dagger(x) A(x)$$

where we have introduced the operators A

$$A \;=\; -\pi_\eta + i\left[\frac{(\lambda-1)}{2}\partial_x \frac{\partial_x \eta}{\rho_s} - \pi\lambda\partial_x \eta^H\right]$$

satisfying the following equal-time commutation relation:

$$\left[A(x), A^\dagger(y)\right] = (1-\lambda)\partial^2_{xy}\frac{\delta(x-y)}{\rho_0} + 2\lambda\partial_x \frac{P}{x-y}$$

Using the equation of motion $\dot{A}(x,t) = i[H, A(x,t)]$, we obtain

$$\left[-i\partial_t + \frac{\lambda-1}{2}\frac{\partial_x\rho_s}{\rho_s}\partial_x - \frac{\lambda-1}{2}\partial_x^2\right](\rho_s A) = -\lambda\pi\rho_s\partial_x(\rho_s A)^H$$

Taking the Hilbert transform of this equation

$$\left[-i\partial_t + \frac{\lambda-1}{2}\frac{\partial_x\rho_s}{\rho_s}\partial_x - \frac{\lambda-1}{2}\partial_x^2\right](\rho_s A)^H = \lambda\pi\rho_s\partial_x(\rho_s A)$$

Defining the fields

$$\Phi_s^\pm = \rho_s^H \pm i\rho_s,\; \phi^\pm = (\rho_s A)^H \pm i(\rho_s A)$$

we find that $\phi^\pm$ satisfies

$$\left\{i\partial_t - \left[\lambda\pi\Phi_s^\pm + \frac{q(\lambda-1)}{2}\mathcal{P}\cot\left(\frac{qx}{2}+\varphi\right)\right]\partial_x + \frac{\lambda-1}{2}\partial_x^2\right\}\phi^\pm = 0$$

(Note: one can interpret the field ϕ as a fluctuation around the conformal field Φ_s.)

Solving previous equation for the semiclassical solution, we obtain the following results:
-the operator A is given by

$$A = \frac{2\pi}{L} \sum_{n,s} e^{i\omega_n t} f_{n,s}(x) \left[\theta(\omega_n) a_{n,s} + \theta(-\omega_n) a^\dagger_{n,s} \right]$$

where the operators $a_{n,s}$ satisfy

$$[a_{n,s}, a^\dagger_{m,s'}] = |\omega_n| L/\pi \delta_{nm} \delta_{ss'}$$

and the functions $f_{n,s}$ are orthonormalized

$$\int_{-L/2}^{L/2} dx \rho_s(x) f^*_{n,s}(x) f_{m,s'}(x) = \frac{L}{2\pi} \delta_{nm} \delta_{s,s'}$$

-the Hamiltonian up to quadratic terms is given by

$$H = E_0 + \frac{\pi}{L} \sum_{n,s} a^\dagger_{n,s} a_{n,s} + \sum_{n,s} \theta(-\omega_n) |\omega_n|$$

λ	ρ_s	$f_{n,\pm}$		ω_n
$\lambda<1$	$\frac{q(1-\lambda)\coth(t/2)}{2\pi\lambda}\frac{1-\cos(qx+2\varphi)}{\cosh t-\cos(qx+2\varphi)}$	$\sqrt{\frac{\lambda(k_0+q)(k_n+q)}{4(1-\lambda)k_0k_n(2k_0+q)}}\left(1-\frac{k_ne^{\pm i(qx+2\varphi)}}{k_n+q}\right)\left(1-\frac{k_0e^{\mp i(qx+2\varphi)}}{k_0+q}\right)\frac{e^{\pm i(k_n-k_0)x}}{1-\cos(qx+2\varphi)}$	$k_n>k_0$	$\frac{1-\lambda}{2}(k_n+k_0+q)(k_n-k_0)$
	$\frac{1-\lambda}{\lambda\pi b}\frac{x^2}{x^2+b^2}$	$\sqrt{\frac{\lambda}{2k_0(1-\lambda)}}\left(1\pm\frac{i}{k_nx}\right)\left(1\mp\frac{i}{k_0x}\right)e^{\pm i(k_n-k_0)x}$	$k_n>k_0$	$\frac{1-\lambda}{2}\left(k_n^2-k_0^2\right)$
	ρ_0	$\frac{1}{\sqrt{2\pi\rho_0}}e^{\pm i(k_n-k_0)x}$	$k_n>k_0$	$\frac{1-\lambda}{2}\left(k_n^2-k_0^2\right)$
$\lambda>1$	$\frac{k(\lambda-1)}{2\pi\lambda}\frac{\sinh t}{\cosh t-\cos kx}$	$\sqrt{\frac{\lambda}{2k_0(\lambda-1)\left(1-e^{-2t}\right)}}\left(1-e^{-t}\,e^{\mp 2ik_0x}\right)e^{\pm i(k_n+k_0)x}$	$k_n>-k_0$	$\frac{\lambda-1}{2}\left(k_0^2-k_n^2\right)$
	$\frac{\lambda-1}{\lambda\pi}\frac{b}{x^2+b^2}$	$\sqrt{\frac{\lambda}{2b(\lambda-1)}}(x\mp ib)\,e^{\pm ik_nx}$	$k_n>0$	$-\frac{\lambda-1}{2}k_n^2$
	ρ_0	$\frac{1}{\sqrt{2\pi\rho_0}}e^{\pm i(k_n+k_0)x}$	$k_n>-k_0$	$\frac{\lambda-1}{2}\left(k_0^2-k_n^2\right)$

Table 1: Excitations around BPS solutions

Discussion and interpretation of the results

About the model

- Why free model?
  Under coordinate reparametrization and field rescaling

$$x = \frac{x'}{\sinh t'}, \quad t = \tanh t', \quad \rho(x,t) = \rho(x',t') \cosh t'$$

  the kinetic term induces the harmonic potential (Avan, Jevicki)

$$\int dxdt \frac{\left(\partial_x^{-1}\dot{\rho}\right)^2}{\rho} = \int dx'dt' \left[\frac{\left(\partial_{x'}^{-1}\dot{\rho}'\right)^2}{\rho'} + x'^2 \rho'(x',t') \right]$$

  and other terms in the Lagrangian remain invariant and therefore all three matrix models have background independence. This property enables us to concentrate the discussion on the free models.

- Interpretation of $\mathcal{P}\cot$ term.
  It was shown that adding the term $(1-\lambda)/(x-z)$ into the effective potential was equivalent to the extraction of the prefactor $\prod_i (x_i - z)^{1-\lambda}$ from the wave function of QM Hamiltonian. This equivalence enables us to associate a quasi-particle located at z with the prefactor of the wave function. Consequently, the additional term $\mathcal{P}\cot(qx/2+\varphi)$ is associated with the prefactor describing M equidistant quasi-particles.

- Compact support.
  Solitons on the compact support are of the same shape as solitons in the Sutherland model, thus reflecting the fact that the two models are interrelated via the periodicity condition.

About the solutions

- $\lambda \leftrightarrow 1/\lambda$ duality.
  There exists a simple relation between systems with $\lambda < 1$ and those with $\lambda > 1$. By substituting $\lambda\rho(x) = \alpha - m(x)$ into Bogomol'nyi

eq. for $\lambda > 1$ (without the term $\mathcal{P}\cot$) and by inserting explicit forms of the solutions for the term ρ^H/ρ, we find that the field m satisfies Bogomol'nyi eq. for $\lambda' = 1/\lambda < 1$ (with the term $\mathcal{P}\cot$). This agrees with the result obtained by Minahan and Polychronakos in the k-space ($\rho_k \to -m_k/\lambda$).

- Giant gravitons?
  The soliton solutions we have found in the collective-field formulation of the free matrix model correspond to the particle and hole states in the system of nonrelativistic fermions (Jevicki). Owing to the $su(1,1)$ dynamical symmetry, the eigenstates of the QM Hamiltonian can be represented as generalized coherent states of the same Hamiltonian with the additional harmonic potential interaction between fermions (AFF; Perelomov). The particle and hole states in the system of fermions with the harmonic potential interaction correspond to the giant gravitons of a $1/2$ BPS sector of $N = 4$ SYM (Corley, Jevicki, Ramgoolam; Berenstein; LLM). Therefore, our solutions correspond to the coherent states of the matrix model with the harmonic potential, i.e. to the quasi-classical CFT duals of the giant gravitons in AdS constructed by Caldarelli and Silva. The nonexistence of the quasi-classical CFT dual of the single giant graviton on the sphere S^5 is reflected throught the fact that the soliton with $M = 1$ in the $\lambda > 1$ case is non-normalizable since M must exceed N.

Duality-based matrix model

A generalization of the hermitian matrix model defined by the Hamiltonian

$$H(x,z) = \sum_{i=1}^{N}\frac{p_i^2}{2} + \frac{1}{2}\sum_{i\neq j}^{N}\frac{\lambda(\lambda-1)}{(x_i-x_j)^2} + \frac{1}{2}\sum_{i,\alpha}^{N,M}\frac{(\kappa+\lambda)(\kappa-1)}{(x_i-Z_\alpha)^2} +$$
$$+ \frac{\lambda}{\kappa}\left[\sum_{\alpha=1}^{M}\frac{p_\alpha^2}{2} + \frac{1}{2}\sum_{\alpha\neq\beta}^{M}\frac{\kappa^2/\lambda\left(\kappa^2/\lambda-1\right)}{(Z_\alpha-Z_\beta)^2}\right]$$

For $\lambda = 1/2$ this model arises from the decomposition of the hermitian matrix into the sum of symmetric and antisymmetric matrix. Transformation into the hydrodynamic formulation for $\kappa = 1$, results in the hermitian collective-field Hamiltonian

$$H = \frac{1}{2}\int dx\rho(x)(\partial_x\pi_\rho(x))^2 + \frac{\lambda}{2}\int dx m(x)(\partial_x\pi_m(x))^2 +$$
$$+ \int dx\frac{\rho(x)}{2}\left[\frac{\lambda-1}{2}\frac{\partial_x\rho(x)}{\rho(x)} + \fint dy\frac{\lambda\rho(y)}{x-y} + \fint dy\frac{m(y)}{x-y}\right]^2 +$$
$$+ \int dx\frac{m(x)}{2\lambda}\left[\frac{1-\lambda}{2}\frac{\partial_x m(x)}{m(x)} + \fint dy\frac{m(y)}{x-y} + \fint dy\frac{\lambda\rho(y)}{x-y}\right]^2 -$$
$$- \frac{\lambda}{2}\int dx\rho(x)\partial_x\frac{P}{x-y}\Big|_{x=y} - \frac{1}{2}\int dx m(x)\partial_x\frac{P}{x-y}\Big|_{x=y}$$

The semiclassical solutions of two coupled Bogomol'nyi equations,

$$(\lambda-1)\partial_x\rho - 2\pi\rho(\lambda\rho^H + m^H) = 0$$
$$(1-\lambda)\partial_x m - 2\pi m(\lambda\rho^H + m^H) = 0$$

Based on the duality, we make an ansatz $m^H = -\lambda\alpha\rho^H/\rho$.

$$(\lambda-1)\partial_x\rho - 2\lambda\pi\rho\rho^H + 2\lambda\alpha\pi\rho^H = 0$$

Again, the field $\Phi = \rho^H + i\rho$ which satisfies the Riccati equation:

$$\partial_x \Phi = \frac{\lambda\pi}{\lambda - 1}\Phi^2 - i\frac{2\lambda\pi\alpha}{\lambda - 1}\Phi + \frac{\lambda\pi\rho_0}{\lambda - 1}(\rho_0 - 2\alpha)$$

The general solution of this equation constructed from the constant solution $\Phi = i\rho_0$ is

$$\Phi(x) = i\rho_0 - \frac{\lambda - 1}{\lambda\pi}\frac{iqce^{iqx}}{1 + ce^{iqx}},\ q = \frac{2\lambda\pi(\alpha - \rho_0)}{1 - \lambda} > 0$$

The solutions for ρ and m $(c = e^{i\phi - u - v},\ |c| < 1)$ are

$$\rho(x) = \alpha\frac{\cosh(u - v) + \cos(qx + \phi)}{\cosh(u + v) + \cos(qx + \phi)},\ m(x) = \frac{\tilde{c}}{\rho(x)}$$

$$q = \frac{4\lambda\pi\alpha}{1 - \lambda}\frac{\sinh u \sinh v}{\sinh(u + v)},\ \frac{\tilde{c}}{\lambda\alpha^2} = \frac{\sinh(u - v)}{\sinh(u + v)},\ u > v > 0$$

Taking $\phi = \pi,\ \sinh(u/2 - v/2) = aq/2,\ \sinh(u/2 + v/2) = bq/2,\ b > 0$, and the limit $q \longrightarrow 0$, we obtain the one-soliton solution

$$\rho(x) = \alpha\frac{x^2 + a^2}{x^2 + b^2},\ m(x) = \frac{\lambda\alpha^2 a}{b\rho(x)},\ a^2 = b^2 + \frac{\lambda - 1}{\lambda\pi\alpha}b$$

The singular limit

Taking the limit $u - v = 2\epsilon \to 0$

$$\rho(x) = \alpha \frac{\cos^2(\frac{qx+\phi}{2})}{\sinh^2 v + \cos^2(\frac{qx+\phi}{2})}, \ \alpha = \frac{(1-\lambda)q}{2\lambda\pi} \coth v$$

$$m(x) = (1-\lambda) \sum_{i=-\infty}^{\infty} \delta(x - x_i), \ x_i = \frac{(2i+1)\pi - \phi}{q}$$

and

$$\rho(x) m(x) \to 0$$