Towards twisted standard model in Moyal space-time

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Plan of the talk

- ◇ Motivations - quantum gravity and space time geometry

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- ◇ new developements poincare invariance, drinfeld twist,

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- ◇ Conclusions
- ◇ Based on: hep-th/0406125,0410067(JHEP),0608179(Phys.Rev D)+ 0706.1259 + 0708.0069 + ongoing work

Acknowledgements: Collaborations with Balachandran, Sachin Vaidya, Giorgio Immirzi, Seckin, Gianpiero Mangano, Alexander Pinzul, Babar Quereshi....

Motivations.....

◇ Quantum gravity -at Planck length - folklore- must have - noncommutative geometric structure - limit of classical gravity - emerge - commutative geometry of spacetime we know. Just like:

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- Expectation:

$$\lim_{Planck\ length \longrightarrow 0} \quad Non\ commutative\ geometry$$
$$=$$
$$Commutative\ Geometry$$

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- ◇ Any attempt to localise events to lengths close to Plancklength will bring in enormous energy and eventually lead to blackholes being created. This will distort the local geometry so much that quantum effects would be overwhelming.
- ◇ The above arguments have been posed in two independent places. (1) Sergio Doplicher's paper. (2)Podles lectures on quantum groups - where it is mentioned that Nahm has posed the questions and the need to go beyond conventional ideas of geometries.

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- The above is from "On the hypotheses which lie at the bases of geometry", Bernhard Riemann, 1854 (from the translation by W K Clifford).

QFT in Moyal spacetimes...

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- This can be understood by the introduction of star product rule in the algebra of functions on R^4. The multiplication map of algebra of functions (*on Moyal plane*) $\mathcal{A}_\theta(R^4)$ is $f \; * \; g \;=\; m_\theta(f \otimes g) \;=\; m_0(F_\theta(f \otimes g))$

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- In commutative spacetime we have pointwise multiplication.

QFT in Moyal....

◇ Consider the scalar field theory on the GM plane with the Lagrangian (density)

$$\mathcal{L}_* = \frac{1}{2}\partial_\mu \Phi * \partial^\mu \Phi - \frac{1}{2}m^2 \Phi * \Phi - \frac{\lambda}{4!}\Phi * \Phi * \Phi * \Phi \,,$$

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◇ Singular $\theta \to 0$ limit makes the theory unsuitable as an effective theory.

Gauge theories...

◇ Conventional Gauge transformations will not close with the new multiplication map given as star product. For this one introduces star gauge transformations: Under star gauge transformation
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- The NC field strength
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu * A_\nu - A_\nu * A_\mu)$
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  under the star gauge transformation.
- Since gauge transformations are introduced in this way there is no way to get gauge groups other than $U(N)$. Infact there is no standard model unless we extend. Charges of $U(1)_{EM}$ are also rigidly fixed.

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- For example the field strength $F_{\mu\nu}$ is expanded as:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ &- \frac{1}{2}\theta^{\rho\gamma}(\partial_\rho A_\mu \partial_\gamma A_\nu - \partial_\rho A_\nu \partial_\gamma A_\mu) + \mathcal{O}(\theta^2) \end{aligned}$$

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- Phenomenological consequences have been worked out. We will not elaborate more on this approach.

New developements...

- The assumption that noncommutativity breaks in general Lorentz invariance is not completely correct. We will show Poincare group algebra acts on the $\mathcal{A}_\theta(R^4)$ Moyal plane if the coproduct is deformed. This is interesting and makes the situation better because while considering field theories on NC space one uses the representation theory of Poincare group without any justification. This will happen for space-space as well as space-time noncommutativity JHEP 0410, 72, 0411, 68.

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- This leads to some interesting results like violation of exclusion principle, pauli-pairs, no uv-ir mixing,.... etc
- This can help in putting experimental bounds on noncommutativity parameter.

poincare covariance....

- If V is a vectorspace and G is a group, $\rho(g)$ a representation then we have the action of the group as: $V \longrightarrow \rho(g)V$.

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- In the theory of Hopf algebra the action of $\mathcal{G}$ is obtained using the coproduct which is homomorphism from $\mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$

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- The choices of coproducts are not all equivalent. For example the IRR's that occur in $\rho \otimes \rho$ and the CG coefficients depend on Δ. This is well known in quantum groups.

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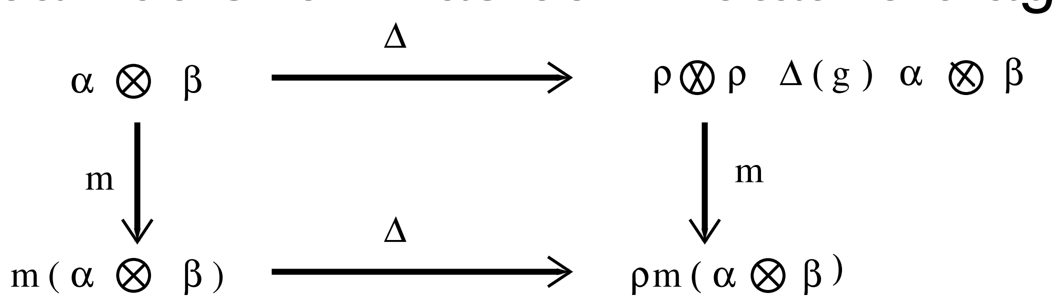
- The above can be shown as commutative diagram!



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- If such a coproduct Δ exists then G acts as an automorphism on V.

poincare covariance....

◇ Indeed such a twisted coproduct$_{\text{Drinfeld}}$ for Moyal space is:

$$\Delta_\theta(g) \;=\; \hat{F}_\theta^{-1}(g \otimes g)\hat{F}_\theta$$

where $\hat{F}_\theta \;=\; e^{-\frac{1}{2}\; P_\mu \otimes \theta^{\mu\nu}\; P_\nu}$, P_μ is the generator of translations.

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- It is easy to check that the coproduct is compatible with the multiplication map.

$$m_\theta(\rho\otimes\rho)\Delta_\theta(g)(\alpha\otimes\beta) \;=\; m_0\left[F_\theta(F_\theta^{-1}\rho(g) \otimes \rho(g)\; F_\theta)\alpha \otimes \beta\right]$$

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- Tensor product of Plane waves $e_p(x) \;=\; e^{ip.x}$ under Lorentz transformations go as:

$$e^{\frac{i}{2}(\Lambda p)_\mu \Theta^{\mu\nu}(\Lambda q)_\nu}\; e^{-\frac{i}{2}p_\mu \Theta^{\mu\nu} q_\nu} e_{\Lambda p} \otimes e_{\Lambda q}$$

Twisting statistics...

◇ For $\theta^{\mu\nu} = 0$ statistics is imposed on the two-particle sector by working with the (a)symmetrized tensor product $\mathcal{A}_0(\mathbb{R}^4) \otimes_{s,a} \mathcal{A}_0(\mathbb{R}^4)$.

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- We are forced to twist statistics also.

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- The tensor product $\mathcal{A}_\theta(\mathbb{R}^4) \otimes_{s_\theta, a_\theta} \mathcal{A}_\theta(\mathbb{R}^4)$ with twisted (a)symmetrization is:

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◇ Like in standard QM, statistics is superselected and all observables commute with τ_θ.

scalar field......

◇ If a scalar field has Fourier expansion as:

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- But on $\phi \ \otimes \ \chi$, twisted Lorentz transformations act as:

$$\Delta_\theta(\Lambda)(\phi \otimes \chi)(p,q) = F_\theta(\Lambda^{-1}p, \Lambda^{-1}q) F_\theta^{-1}(p,q)\phi(\Lambda^{-1}p)\chi(\Lambda^{-1}q)$$

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◇ where $F_\theta(p,q) \;=\; e^{-\frac{i}{2}p\,\cdot\,\theta\,\cdot\,q}$.

exclusion principle......

- We will now show that for the scalar field ϕ we have new deformed operator relations:

$$a(p)a(q) \;=\; \eta\, F_\theta^{-2}(q,p)a(q)a(p)$$

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◇ If we suppose

$$a(p)a(q) \; = \; G_\theta(p,q)a(q)a(p)$$

then

$$U(\Lambda) \; G_\theta(p,q) \; U(\Lambda)^{-1} \; = \; G_\theta(p,q)$$

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- The above was known as Faddeev - Zamolodchikov algebra in 2D integrable models. For fermions(bosons), in the limit of $\theta = 0$, we have $\eta = -1(+1)$.

new exclusion principle......

- A single particle state is given by $|\alpha\rangle = \int Dp\alpha(p)\, a^\dagger_p\, |0\rangle$. We can ask whether two particle symmetric state

$$|\alpha, \alpha\rangle = \int Dp Dq \alpha(p)\alpha(q) a^\dagger_p a^\dagger_q\, |0\rangle$$

is permitted - violating pauli statistics.

new exclusion principle......

- A single particle state is given by $|\alpha\rangle = \int Dp\alpha(p)\, a_p^\dagger\, |0\rangle$. We can ask whether two particle symmetric state

$$|\alpha,\alpha\rangle = \int DpDq\alpha(p)\alpha(q)a_p^\dagger a_q^\dagger\, |0\rangle$$

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- pauli pairs- we can also show even more intriguing features like two particle states of certain types are not allowed. These are generalisations of two particle symmetric states for fermions bal,giorgio,trg,vaidya.

uv/ir mixing,....

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- Given the single particle annihilation operators a_p we can define operators c_p obeying standard relations.

$$a_p \;=\; c_p \, e^{\frac{i}{2} p_\mu \Theta^{\mu\nu} \, P_\nu}$$

Here P_μ is the translations generator.

$$P_\mu \;=\; \int d\mu(p)\, p_\mu \, a^\dagger(p) a(p)$$

uv/ir mixing,...

◇ The interaction Hamiltonian is:

$$H_I(t) \;=\; \lambda \int dx : \phi_*^n :$$

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- to order λ we will have

$$: \phi * \phi * \phi \cdots \phi : \;=\; : a(p_1) a(p_2) ... a(p_n) :$$

which simplifies to

$$: c(p_1) c(p_2) ... c(p_n) : \; e_{p_1+p_2+\cdots p_n}(x)\; e^{\frac{i}{2}(p_1+p_2+\cdots p_n)\circ\Theta\circ P}$$

uv/ir mixing,...

- And using 4-momentum conservation we get
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- But the scattering amplitudes will depend on θ as the in and out states are changed.
- There is an easier way to understand the above features as well as introduce diffeos and gauge symmetry using a novel commutative algebraic substructure inside $\mathcal{A}_\theta(R^4)$.

The commutative algebra $\mathcal{A}_0(R^4)$

◇ Let us see how we can define diffeomorphisms and gauge symmetries in this framework. But the coproduct again should be changed to be compatible with multiplication. Wess etal, But we will adopt a novel way.

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- where $x^L_\mu \, \alpha = x_\mu * \alpha$ and $x^R_\mu \, \alpha = \alpha * x_\mu$.
- It is easy to see

$$[x^c_\mu, x^c_\nu] = 0.$$

This simply means x^c_μ form a basis for commutative algebra $A_0(R^4)$. One can define Poincare group of generators using x^c_μ as

$$M_{\mu\nu} = x^c_\mu \, p_\nu - x^c_\nu \, p_\mu \, , p_\mu = -i\partial_\mu$$

Diffeomorphism and gauge invariance

◇ We get modified Leibnitz rule:

$$
\begin{aligned}
M_{\mu\nu}(\alpha * \beta) &= M_{\mu\nu}\alpha * \beta \; + \; \alpha * M_{\mu\nu}\beta \\
&- \frac{1}{2}[(p.\theta)_\mu \alpha * p_\nu \beta - (p_\nu \alpha * (p.\theta)_\mu \beta - \mu \leftrightarrow \nu]
\end{aligned}
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- $M_{\mu\nu}$ is a particular vector field. This can be extended to general vector fields $v = v^\mu(x^c)\partial_\mu$.
- These generate the diffeomorphisms on the Moyal spacetime.

Diffeomorphism and gauge invariance

- Consider covariant derivative $D_\mu = \partial_\mu + \Gamma_\mu + \omega_\mu$. If we assume the framefields e^a_μ are dependent only on x^c then pure gravity without matter can be treated as in commutative spacetimes.

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- Gauge fields A_λ transform as one-forms under diffeomorphisms for $\theta^{\mu\nu} = 0$. For $\theta^{\mu\nu} \neq 0$, the vector fields v^μ generating diffeomorphisms depend on x^c.
- If a diffeomorphism acts on A_λ in a conventional way and $A_\lambda, \delta A_\lambda$ are to depend on just one combination of noncommutative coordinates, then A_λ can depend only on x^c.

Diffeomorphism and gauge invariance

- Twisted coproducts for diffeos are needed to maintain them as symmetries in gravity. But with gravity and gauge fields present, the group of importance is not just $\mathcal{D}_0(\mathbb{R}^4)$, but its semi-direct product $\mathcal{G} \ltimes \mathcal{D}_0(\mathbb{R}^4)$.

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- it is natural to keep $\mathcal{G} \ltimes \mathcal{D}_0(\mathbb{R}^4)$ for $\theta^{\mu\nu} \neq 0$. $\mathcal{D}_0(\mathbb{R}^4)$ perform diffeomorphisms. We require elements of $\mathcal{G}$ are constructed from the elements of the algebra generated by x^c and the group $\mathcal{G}$ is independent of $\theta^{\mu\nu}$.

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- The conclusion is that pure gravity and gauge sectors are unaffected by noncommutativity.

Diffeomorphism and gauge invariance

- In the standard approach to noncommutative gauge groups covariant derivatives act with the $*$ -product it is possible to have only particular representations of $U(N)$ gauge groups or use enveloping algebras. There is no such limitation now where the gauge group.

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- ◇ In quantum Hall effect, the algebra of observables is $\mathcal{A}_\theta(\mathbb{R}^2) \otimes \mathcal{A}_\theta(\mathbb{R}^2)$. Here too covariant derivatives of the $U(1)$ electromagnetism do act in the same way and not with a $*$ product.

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- In Wess et al.,the covariant derivative D^*_μ acts with a $*$ -product. Hence:

$$\mathcal{D}^*_\mu = D^*_\mu e^{-\frac{i}{2} ad \overleftarrow{\partial}_\lambda \theta^{\lambda\rho} \overrightarrow{\partial}_\rho}; \mathcal{D}^*_\mu * \alpha = D^*_\mu \alpha$$

Gauge group on matter fields

◇ Fields transform non-trivially under $\mathcal{G}$ or “global” group G are modules over $\mathcal{A}_\theta(\mathbb{R}^4)$. If a d-dimensional representation of G is involved, they can be elements of $\mathcal{A}_\theta(\mathbb{R}^4) \otimes \mathbb{C}^d$.

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- We need the action of gauge transformations on these modules compatibly with the $*$-product.
- We should form gauge scalars out of elements of $\mathcal{A}_\theta(\mathbb{R}^4) \otimes \mathbb{C}^d$ and their adjoints. We can do these consistently only if the gauge group also has a twisted coproduct.

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- We need the action of gauge transformations on these modules compatibly with the $*$-product.
- We should form gauge scalars out of elements of $\mathcal{A}_\theta(\mathbb{R}^4) \otimes \mathbb{C}^d$ and their adjoints. We can do these consistently only if the gauge group also has a twisted coproduct.
- The twisted coproduct on $\mathcal{G}$ is,

$$\Delta_\theta(g(x^c)) = F_\theta^{-1}[g(x^c) \otimes g(x^c)]F_\theta,$$

and is compatible with the $*$-multiplication.

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◇ This twisted coproduct $\Delta_\theta(g(\hat{x}^c)$ preserves the semi-direct product structure $\mathcal{G} \ltimes \mathcal{D}_0(\mathbb{R}^4)$.

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- Next we need covariant derivatives consistently defined to complete the program.
- We already saw the twisted commutation relations:

$$
\begin{aligned}
a(p)a(q) &= e^{ip\wedge q}a(q)a(p), \\
a(p)a^\dagger(q) &= e^{-ip\wedge q}a^\dagger(q)a(p) + 2p_0\delta^{(3)}(p-q),
\end{aligned}
$$

Dressing transformation..

- Now $a(p), a^\dagger(p)$ can be realized in terms of untwisted Fock space operators $c(p), c^\dagger(p)$ by the “dressing transformation" grosse,zamolodchikov,faddeev

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\begin{aligned}
a(p) &= c(p)e^{-\frac{i}{2}p\wedge P}, \quad a^\dagger(p) = c^\dagger(q)e^{\frac{i}{2}p\wedge P}, \text{where} \\
P_\mu &= \int d\mu(q)q_\mu[a^\dagger(q)a(q)] = \text{total momentum operator.}
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- Then $\phi(x)$ may be written in terms of commutative fields ϕ^c as

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◇ Then $\phi(x)$ may be written in terms of commutative fields ϕ^c as

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◇ If $\phi_1, \phi_2, \cdots \phi_n$ are quantum fields, $\phi_i(x) = \phi_i^c e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}(x)$,

Covariant derivatives,...

◇ then

$$(\phi_1 * \phi_2 * \cdots \phi_n)(x) = (\phi_1^c \phi_2^c \cdots \phi_n^c) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}(x)$$

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- For example Interaction Hamiltonian density is:

$$\mathcal{H}_{I\theta} = \mathcal{H}_{I0}\, e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}$$

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◇ The covariant derivative should transport consistently with the statistics and gauge transformations and the natural choice is:

$$D_\mu \phi = ((D_\mu)^c \phi^c) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}$$

Covariant derivatives,...

◇ It is easy to check:

$$[D_\mu, D_\nu]\varphi = \Big([D^c_\mu, D^c_\nu]\varphi^c\Big)e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} = \Big(F^c_{\mu\nu}\varphi^c\Big)e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}.$$

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- We can also write:

$$D_\mu\varphi \; = \; \Big(D^c_\mu e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}\Big) \star \Big(\varphi^c e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}\Big).$$

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◇ As $F^c_{\mu\nu}$ is the standard $\theta^{\mu\nu} = 0$ curvature, gauge field is that of commutative space-time and transforms covariantly under gauge transformations. We can use it to construct the Hamiltonian.

Gauge theory on moyal space-time...

◇ The interaction Hamiltonian density for pure gauge fields is:

$$\mathcal{H}^{G}_{I\theta} = \mathcal{H}^{G}_{I0}.$$

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- But when we have both matter and gauge fields the interaction Hamiltonian density:

$$\mathcal{H}_{I\theta} = \mathcal{H}^{M,G}_{I\theta} + \mathcal{H}^{G}_{I\theta},$$

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- In QCD_θ, we have $\mathcal{H}^{SU(3)}_{I\theta} = \mathcal{H}^{SU(3)}_{I0} \neq 0$, so that

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- Lastly we look for Standard model$_\theta$ with spontaneous symmetry breakdown.

Higgs$_\theta$ mechanism

◇ We start with Higgs potential

$$V(\phi) = \lambda(\phi^\dagger * \phi - a^2)^2_* = \lambda(\phi_c^\dagger \phi_c - a^2)\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

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- We assume the breaking $G \longrightarrow H$. In vacuum

$$\begin{aligned} \langle \phi_c \rangle &= \phi^0, \quad \phi^{0\dagger}\, \phi^0 = a^2, \\ h\, \phi^0 &= \phi^0, h \in H \end{aligned}$$

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◇ The vacuum manifold is

$$\phi = g\, \phi^0, \quad g \in G, \; and \quad (gh)\, \phi^0 = g\, \phi^0$$

Mass of the gauge boson

- The gauge field acquires mass and is given by the term:

$$M = (D_\mu \phi)^\dagger * (D^\mu \phi) = [(D^c_\mu \phi_c)^\dagger (D^{\mu c} \phi_c)] e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}$$

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- If a gauge transformation is performed from $A^c_\mu \longrightarrow B^c_\mu$ where $B^c_\mu = g^\dagger\, D^c_\mu\, g$, then

$$M = {\phi^c}^\dagger{}_\alpha ({B^c_\mu}^\dagger B^{\mu c})_{\alpha\beta} \phi^c_\beta$$

Mass of the gauge boson

◇ As usual we write

$$B^c_\mu \;=\; B^{c\,\alpha}_\mu V_\alpha \;+\; B^{c\,i}_\mu S_i$$

Then we get:

$$M \;=\; (D^c_\mu \phi^c)^\dagger (D^{\mu c}\phi^c) \;=\; \phi^{0\dagger} S_i B^i_\mu B^{\mu j} S_j \phi^0 \;+\; \cdots$$

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- This shows gauge fields in the direction of V_α dont acquire mass and only those in the direction of S_i do.

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Then we get:

$$M = (D^{c}_{\mu}\phi^{c})^{\dagger}(D^{\mu c}\phi^{c}) = \phi^{0\dagger} S_{i} B^{i}_{\mu} B^{\mu j} S_{j} \phi^{0} + \cdots$$

- This shows gauge fields in the direction of V_{α} dont acquire mass and only those in the direction of S_{i} do.
- B^{c}_{μ} is the gauge transformation of D^{c}_{μ}. This preserves the pure gauge Hamiltonian $H_{I\theta} = H_{I0}$.

Mass of the gauge boson

- After gauge fixing the Hamiltonian with the mass term is:

$$H_0 \;=\; \int \{\partial \wedge B^c)^2 \;+\; (\partial_0 B^i - \partial^i B_0)^2 + \cdots + M\}$$

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◇ Now M can be expressed as:

$$\int d^3x\ M = \int d^3x\ M_0 \Big(e^{\frac{1}{2}\overleftarrow{\partial_0}\theta^{0i}P_i}\Big)\Big(e^{\frac{1}{2}\overleftarrow{\partial_i}\theta^{0i}P_0}\Big)$$

Mass of the gauge boson

- The last term in the exponential gives 1 and hence we are left with:

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- Hence For $\theta_{0i} = 0$ we have $H_{\theta 0} = H_{00}$.
- But there will be additional interaction terms coming from $H_{I\theta}^{M,G} \neq H_{I0}^{M,G}$.

$e^- - e^-$ scattering

- Define: $x = E/m$ and $t = m^2(\vec{T}\cdot\hat{n}),\ T^i = \theta_{ij}\epsilon^{ijk}$ and $\hat{n}$ the unit vector normal to the plane $\hat{p}_i \Leftrightarrow \hat{p}_f$

$$|\mathcal{F}|^2 = |\mathcal{T}(t,\Theta_M,x)|^2/|\mathcal{T}(0,\Pi/4,x)|^2$$

and we plot $|\mathcal{F}|^2 \Leftrightarrow \Theta_M$.

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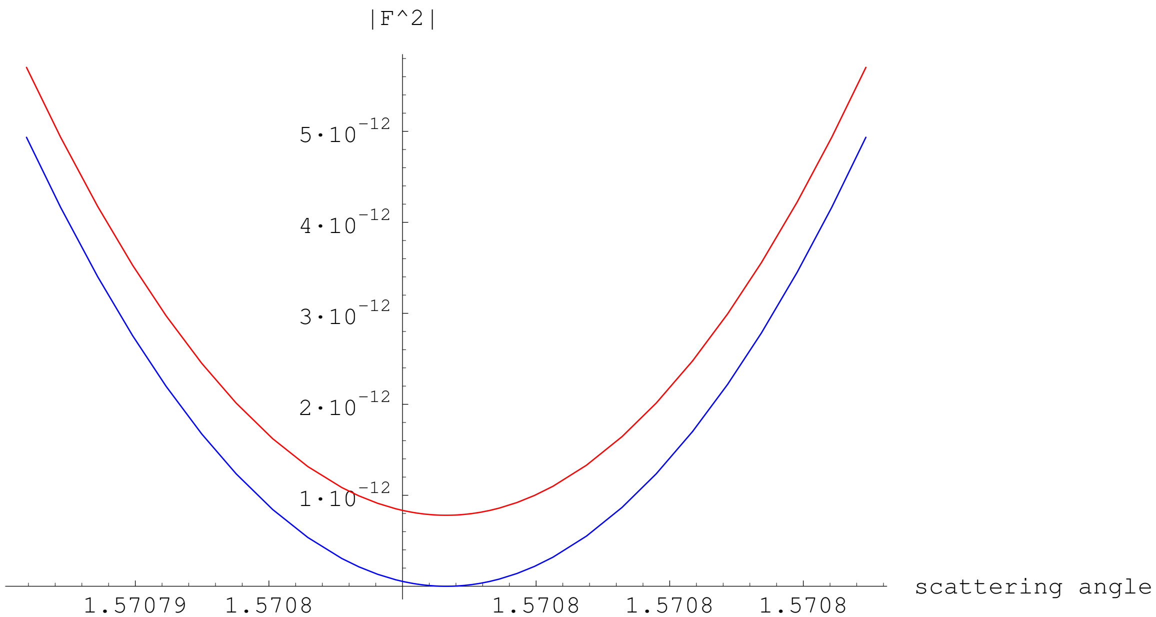

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- We see that NC amplitude does not vanish for $\Theta_M = \pi/2$.