# THERMODYNAMICS OF DARK ENERGY

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# Motivation

Some important questions concerning dark energy (DE):

- Does the DE fluid possess thermal (besides hydrodynamic) properties, such as temperature?
- What is the thermodynamic fate of the Universe dominated by DE?
- Can DE cluster?

- DE (substance of negative pressure) becomes hotter if it undergoes an adiabatic expansion

  J.A.S. Lima, J.S. Alcaniz, PLB 600 (2004)

- Phantom DE violates the null energy condition (NEC: $p+\rho>0$) and hence must have either T<0 or S<0

  Y. Gong, B.Wang, A. Wang, PRD 75 (2007)

  H. Mohseni Sadjadi, PRD 73 (2006)

- T<0 implies that the phantom should be quantized (?) or defining the phantom space to be Euclidean (?)

  P.F. Gonzalez-Diaz , C.L. Siguenza, NPB 697 (2004)

# Outline

- Basic cosmology
- Dark energy  models
- Thomas-Fermi correspondence
- K-essence thermodynamics

# Basic cosmology

- Homogeneity of space

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

- Matter described by of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$$

$u_\mu$ - fluid velocity

$T_{\mu\nu}$- energy-momentum tensor

# Field equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$H \equiv \frac{\dot{a}}{a} \quad \text{expansion rate}$$

$$k \begin{cases} > 0 & \text{closed} \\ = 0 & \text{flat} \\ < 0 & \text{open} \end{cases}$$

$$T^{\mu\nu}{}_{;\nu} = 0 \quad \Rightarrow \quad \rho \propto a^{-3(1+w)} \qquad w = p/\rho$$

## Cosmic fluids with different $w$

- radiation $\quad p_{\mathrm{R}} = \rho_{\mathrm{R}}/3 \quad w = 1/3 \quad \rho \propto a^{-4}$
- matter $\quad p_{\mathrm{M}} = 0 \quad w = 0 \quad \rho \propto a^{-3}$
- vacuum $\quad p_{\Lambda} = -\rho_{\Lambda} \quad w = -1 \quad \rho \propto a^{0}$

- Flatness (k=0):

$$\rho = \rho_{\mathrm{cr}} = 3H^2 / 8\pi G$$ critical density

- Vacuum energy density $\rho_\Lambda = \Lambda / 8\pi G$ is related to the cosmological constant $\Lambda \neq 0 \Rightarrow$ accelerating expansion caused by a negative vacuum energy pressure!
- A new term is coined for a cosmic substance of negative pressure **Dark Energy**
- Comparison of the standard Big Bang model with observations (SN 1a and CMB) require a vacuum energy density of the order

$$\rho_\Lambda \simeq 70\% \, \rho_{\mathrm{cr}}$$

# Dark energy models

- cosmological constant – energy density is constant in time
- k-essence and quintessence – new scalar field – energy density varies with time
- quartessence – the term was coined to describe unified dark matter/dark energy models
- phantom energy – negative pressure exceeds $\rho$ so that the null energy condition is violated, i.e.,

$$p+\rho < 0$$

possible disastrous consequence : Big Rip – decay of all bound systems in finite time

Recent review:

E.J. Copeland, M. Sami, S. Tsujikawa, hep-th/0603057

# Quintessence

P.Ratra, J. Peebles PRD 37 (1988)

Scalar field θ with selfinteraction effectively providing a slow roll inflation for today

$$S = \int d^4x \, \mathcal{L}(X, \theta)$$

$$X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

$$\mathcal{L} = \frac{1}{2}X - V(\theta)$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}} = \theta_{,\mu}\theta_{,\nu} - \mathcal{L} g_{\mu\nu}$$

Field theory description of a perfect fluid if $X>0$

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$

where

$$u_\mu = \frac{\theta_{,\mu}}{\sqrt{X}}$$

$$p = \mathcal{L} = \frac{1}{2}X - V(\theta)$$

$$\rho = X - \mathcal{L} = \frac{1}{2}X + V(\theta)$$

A suitable choice of *V(θ) y*ields a desired cosmology, or vice versa: from a desired equation of state *p=p(ρ)* one can derive the Lagrangian of the corresponding scalar field theory

# k-essence

C. Armendariz-Picon, V. Mukhanov, P.J. Steinhardt, PRL 85 (2000)

Noncanonical kinetic term

$$S = \int d^4x \, \mathcal{L}(X, \theta) \qquad X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

$$\mathcal{L} = A(\theta)K(X) + V(\theta)$$

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial X}\theta_{,\mu}\theta_{,\nu} - \mathcal{L}g_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$

where

$$u_\mu = \frac{\theta_{,\mu}}{\sqrt{X}}$$

$$p = \mathcal{L}$$

$$\rho = 2X\mathcal{L}_X - \mathcal{L} \qquad \mathcal{L}_X \equiv \frac{\partial \mathcal{L}}{\partial X}$$

Again, a suitable choice of *A(θ), K(X) and V(θ) y*ields a desired cosmology. The reverse is not unique: from a desired equation of state $p=p(\rho)$ one can derive uncountably many k-essence field theories.

# Examples

- Quintessence: *A(θ)*=1/2 , *K(X)*=X

$$p = \mathcal{L} = \frac{1}{2}X - V(\theta) \qquad \rho = \frac{1}{2}X + V(\theta)$$

- Phantom quintessence: *A(θ)*=-1/2, *K(X)*=X

$$p = \mathcal{L} = -\frac{1}{2}X - V(\theta) \qquad \rho = -\frac{1}{2}X + V(\theta)$$

Obviously,

$$p + \rho \leq 0$$

Violation of NEC!

- Tachion condensate $V(\theta)=0$, $K(X)=\sqrt{1-X^2}$

$$p=\mathcal{L}=-A(\theta)\sqrt{1-X^2} \qquad \rho=\frac{A(\theta)}{\sqrt{1-X^2}}$$

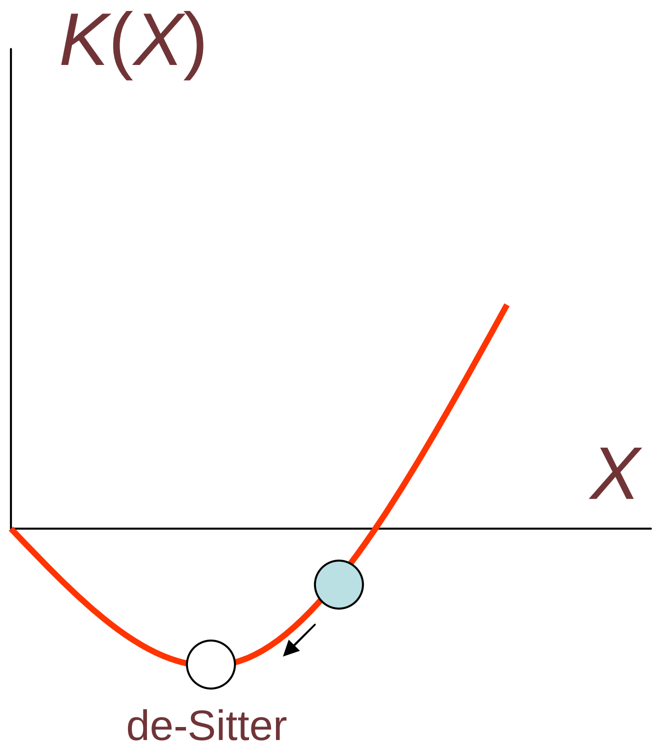
- Kinetic k-essence: $A(\theta)=1$ , $V(\theta)=0$

$$p=\mathcal{L}=K(X) \qquad \rho=2XK_X-K$$

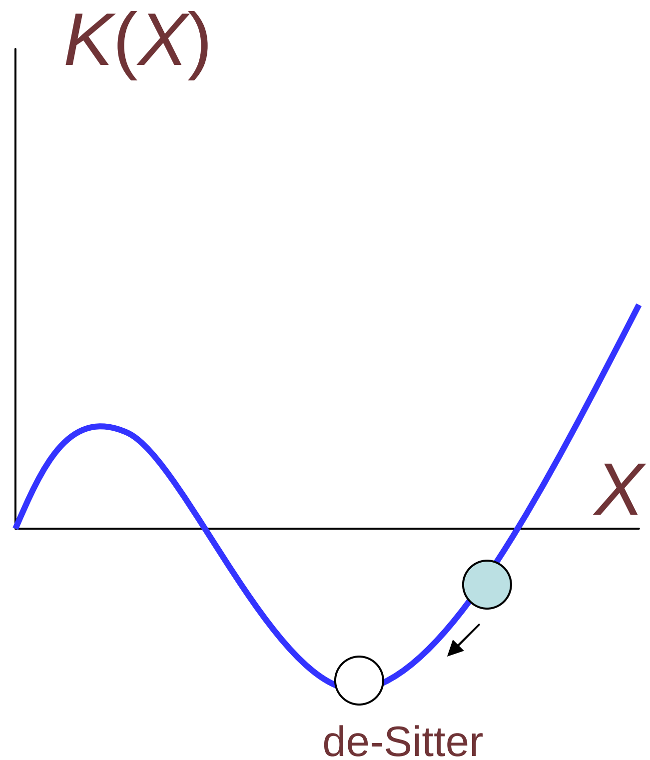
To this class belong the ghost condensate and the scalar Born-Infeld model

# Ghost condensate model

N. Arkani-Hamed et al , JHEP **05** (2004)
R.J. Scherrer, PRL **93** (2004)




or




example

$$K(X) = (X - 1)^2$$

# Quartessence

Example: Chaplygin gas

An exotic fluid with an equation of state

$$p = -\frac{A}{\rho}$$

The first definite model for a dark matter/energy unification

A. Kamenshchik, U. Moschella, V. Pasquier, PLB **511** (2001)
N.B., G.B. Tupper, R.D. Viollier, PLB **535** (2002)
J.C. Fabris, S.V.B. Goncalves, P.E. de Souza, GRG **34** (2002)

# The generalized Chaplygin gas

$$p = -\frac{A}{\rho^{\alpha}} \qquad 0 \leq \alpha \leq 1$$

M.C. Bento, O. Bertolami, and A.A. Sen, PRD **66** (2002)

The term "quartessence" was coined to describe unified dark matter/dark energy models

The Chaplygin gas model is equivalent to (scalar) Dirac-Born-Infeld description of a D-brane:

Nambu-Goto action of a p-brane moving in a *p+2* -dimensional bulk

$$S_{\mathrm{DBI}} = -\sqrt{A}\int d^{p+1}x\,\sqrt{(-1)^p \det(g^{\mathrm{ind}})}$$

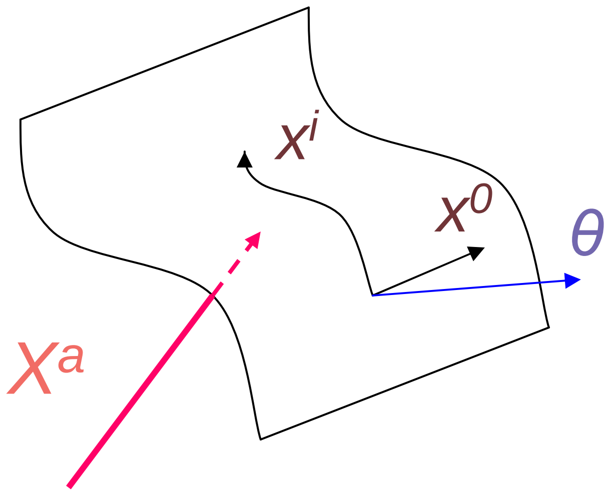
the induced metric (“pull back”) of the bulk metric

$$g^{\mathrm{ind}}_{\mu\nu} = G_{ab}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}$$

Choose the coordinates such that
$X^{\mu} = x^{\mu}$, $\mu=0,..p$, and let the $p$+1-th coordinate
$X^{p+1} \equiv \theta$ be normal to the brane. From now on we set $p$=3. Then

$$G_{\mu\nu} = g_{\mu\nu} \quad \text{for} \quad \mu = 0...3$$

$$G_{\mu 4} = 0 \qquad G_{44} = -1$$



We find a k-essence type of theory

$$S_{\mathrm{DBI}} = -\sqrt{A}\int dx^4 \sqrt{1-X^2} \qquad X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

with

$$\rho = \frac{\sqrt{A}}{\sqrt{1-X^2}}; \qquad p = -\sqrt{A}\sqrt{1-X^2}$$

and hence

$$p = -\frac{A}{\rho}$$

In a homogeneous model the conservation equation yields the density as a function of the scale factor $a$

$$\rho = \sqrt{A + B/a^6}$$

where *B* is an integration constant.The Chaplygin gas thus Interpolates between dust ($\rho \sim a^{-3}$) at large redshifts and a cosmological constant ($\rho \sim A^{1/2}$=const) today and hence yields a correct homogeneous cosmology

# Thomas-Fermi correspondence

Under reasonable assumptions in the cosmological context there exist an equivalence

| Complex scalar field theories (canonical or phantom) | ⟷ | k-essence type of models (canonical or phantom) |
| --- | --- | --- |

Consider

$$\mathcal{L} = \eta g^{\mu\nu}\Phi^*{}_{,\mu}\Phi_{,\nu} - V(|\Phi|^2/m^2) \qquad \Phi = \frac{\phi}{\sqrt{2}}e^{-im\theta}$$

$\eta$=1 canonical; $\eta$=-1 phantom

Thomas-Fermi approximation $\phi_{,\mu} \ll m\phi$

➡ $\Phi_{,\mu} = -i\Phi\,\theta_{,\mu}$

➡ TF Lagrangian $\mathcal{L}_{\mathrm{TF}}/m^4 = XY - U(Y)$

where $X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$ $\quad Y = \eta\dfrac{\phi^2}{2m^2}$ $\quad U(Y) = V(\eta Y)/m^4$

# Applicability of the TF approximation

The approximation $\phi_{,\mu} \ll m\phi$ in homogeneous cosmology means

$$\frac{a}{\phi}\frac{d\phi}{da} \ll \frac{m}{H}$$

Typically $m \simeq \rho_{\rm cr}^{1/4}$ and $H = H_0 a^{-3/2} = \left(\frac{8\pi\rho_{\rm cr}}{3M_{\rm Pl}^2 a^3}\right)^{1/2}$

This yields a requirement $a \gg \left(\rho_{\rm cr}^{1/4}/M_{\rm Pl}\right)^{2/3} \simeq 10^{-20}$

Hence, the TF approximation works extremely well in the matter dominated era $a > a_{\rm eq} \simeq 3\times10^{-4}$

## Equations of motion for $\varphi$ and $\theta$

$$X - \frac{\partial U}{\partial Y} = 0$$

$$(Y g^{\mu\nu} \theta_{,\nu})_{;\mu} = 0$$

## Legendre transformation

$$W(X) + U(Y) = XY$$

$$U_Y \equiv \frac{\partial U}{\partial Y}$$

with $X = U_Y$ and $Y = W_X$

$$W_X \equiv \frac{\partial W}{\partial X}$$

# correspondence

## Complex scalar FT

$$\mathcal{L} = \eta g^{\mu\nu}\Phi^*_{,\mu}\Phi_{,\nu} - V(|\Phi|^2/m^2)$$

$$\Phi = \frac{\phi}{\sqrt{2}} e^{-im\theta}$$

$\Longleftrightarrow$

## Kinetic k-essence FT

$$\mathcal{L} = m^4 W(X)$$

$$X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

## Eqs. of motion

$$(\phi^2 g^{\mu\nu}\theta_{,\nu})_{;\mu} = 0$$

$$g^{\mu\nu}\theta_{,\mu}\theta_{,\nu} - \frac{1}{m^2}\frac{dV}{d|\Phi|^2} = 0$$

## Eq. of motion

$$(W_X g^{\mu\nu}\theta_{,\nu})_{;\mu} = 0$$

## Parametric eq. of state

$$p = m^4 W \qquad \rho = m^4(2XW_X - W)$$

# Current conservation

Klein-Gordon current

$$j^{\mu} = i g^{\mu\nu} (\Phi^{*} \Phi_{,\nu} - \Phi \Phi^{*}_{,\nu})$$

kinetic k-essence current

$$j^{\mu} = 2m^{2} W_{X} g^{\mu\nu} \theta_{,\nu}$$

U(1) symmetry $\Longleftrightarrow$ shift symmetry

$$\Phi \rightarrow e^{-i\alpha} \Phi$$

$$\theta \rightarrow \theta + c$$

# Example: Quartic potential

Scalar field potential

$$V = V_0 \pm m_0^2 \,|\Phi|^2 + \lambda \,|\Phi|^4$$

Kinetic k-essence

$$U(Y) = \frac{1}{2}\left(\eta Y \pm \frac{1}{2\lambda}\right)^2 - \frac{1}{8\lambda^2} \quad \Longleftrightarrow \quad W(X) = \frac{1}{2}\left(\eta X \mp \frac{1}{2\lambda}\right)^2$$

# Example: Chaplygin gas

Scalar field potential

$$V = m^4 \left( \frac{|\Phi|^2}{m^2} + \frac{m^2}{|\Phi|^2} \right)$$

Scalar Born-Infeld FT

$$U(Y) = \eta \left( Y + \frac{1}{Y} \right) \quad \Longleftrightarrow \quad W(X) = -2\sqrt{1 - \eta X}$$

# K-essence thermodynamics

Consider a barotropic fluid with the eq. of state in a parametric form $p = p(X); \qquad \rho = \rho(X)$

Which satisfies the k-essence relation

$$Ts = \rho + p - \mu n$$

Start from

$$d(\rho V) = TdS - pdV$$

If there exist a conserved charge *Q*, with *n=Q/V,* then

$$d\rho = Tds + \mu dn$$

$\mu = \dfrac{\rho + p - Ts}{n}$ Chemical potential $\qquad s = S/V$ Entropy density

$$\Longrightarrow \qquad Ts = \rho + p - \mu n$$

Obviously, if $\mu$=0, a violation of NEC, i.e., $p+\rho<0$ implies either $T<0$ or $S<0$. Hence, for a reasonable thermodynamics we must have $\mu \neq 0$. It follows

$$s = \left.\frac{\partial p}{\partial T}\right|_{\mu} \qquad\qquad n = \left.\frac{\partial p}{\partial \mu}\right|_{T}$$

and

$$p + \rho = T\frac{\partial p}{\partial T} + \mu\frac{\partial p}{\partial \mu}$$

This yields a partial differential equation for $X$

$$T\frac{\partial X}{\partial T} + \mu\frac{\partial X}{\partial \mu} = 2X$$

with a general solution

$$X = \frac{\mu^2}{m^2} f(T/\mu)$$

For a general kinetic k-essence Lagrangian $\mathcal{L}(X)$

the entropy density is

$$s = X\mathcal{L}_X \frac{1}{\mu}\frac{f'}{f}$$

where $\mathcal{L}_X \sim p + \rho > 0$ for the canonical and $\mathcal{L}_X < 0$ for the phantom k-essence.

Now we require $S \geq 0$, with $S=0$ at $T=0$. This implies

$$f'(0) = 0$$ and $$\mu f' > 0 \text{ for } \mathcal{L}_X > 0$$ $$\mu f' < 0 \text{ for } \mathcal{L}_X < 0$$ at $T \neq 0$

Example

$$f = \mathrm{C}_1 \pm \mathrm{C}_2\left(\frac{T}{\mu}\right)^2 \qquad \mathrm{C}_1 \geq 0; \quad \mathrm{C}_2 \geq 0$$

# Chemical potential in a general kinetic k-essence

A field theory described by

$$S = \int d^4x \mathcal{L}(X), \qquad X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

possesses a conserved current $$j^\mu = \frac{2}{m^2}\mathcal{L}_X g^{\mu\nu}\theta_{,\nu}$$

with conserved charge

$$Q = \int_\Sigma j^\mu d\Sigma_\mu = \int_\Sigma n u^\mu d\Sigma_\mu \qquad n = \frac{2}{m}\sqrt{X}\mathcal{L}_X$$

If we choose the hypersurface $\Sigma$ at constant $t$ then

$$Q = \frac{2}{m}\int_V dV g^{0\nu}\theta_{,\nu}\mathcal{L}_X = \frac{1}{m}\int_V dV \frac{\partial \mathcal{L}}{\partial \theta_{,0}}$$

The chemical potential $\mu$ associated with the conserved charge $Q$ is introduced via the grandcanonical partition function

$$Z = \mathrm{Tr}\ \exp - \beta(\hat{H} - \mu \hat{Q})$$

$$= \int\limits_{\text{periodic}} [d\pi] \int [d\theta] \exp \int_0^\beta d\tau \int dV \left( i\pi \frac{\partial \theta}{\partial \tau} - \mathcal{H}(\pi, \theta_{,i}) + \frac{\mu}{m}\pi \right)$$

$\tau = it$ Euclidean time $\beta = 1/T$ inverse temperature

The Hamiltonian density is defined as a Legendre transformation

$$\mathcal{H}(\pi, \theta_{,i}) = \pi\theta_{,0} - \mathcal{L}(\theta_{,0}, \theta_{,i})$$

with

$$\pi = \frac{\partial \mathcal{L}}{\partial \theta_{,0}} \qquad \theta_{,0} = \frac{\partial \mathcal{H}}{\partial \pi}$$

Functional integration over $\pi$ gives

$$Z = \int_{\text{periodic}} [d\theta] \exp -\int_0^\beta d\tau \int dV \mathcal{L}_E(\theta, \mu)$$

The saddle point approximation yields the Euclidean Lagrangian

$$\mathcal{L}_E(\theta, \mu) = -\mathcal{L}\left(\theta_{,0} = i\frac{\partial\theta}{\partial\tau} + \frac{\mu}{m}, \theta_{,i}\right)$$

Hence, the effective Lagrangian is obtained from $\mathcal{L}(X)$ by a replacement

$$\theta_{,\nu} \rightarrow \theta_{,\nu} + \frac{\mu}{m}\delta_\nu^0$$

Note similarity with the prescription for a complex scalar FT

$$\frac{\partial\Phi^*}{\partial\tau} \rightarrow \frac{\partial\Phi^*}{\partial\tau} + \mu\Phi^* \qquad \frac{\partial\Phi}{\partial\tau} \rightarrow \frac{\partial\Phi}{\partial\tau} - \mu\Phi$$

In the comoving frame $u^{\mu} = \delta_0^{\mu} / \sqrt{g_{00}}$. Comparing this with $u^{\mu} = g^{\mu\nu}\theta_{,\nu} / \sqrt{X}$ , we conclude that $\theta$ is a function of $t$ only. Then

$$X = g^{00}(\theta_{,0} + \mu/m)^2$$

This, compared with the general solution

$$X = \frac{\mu^2}{m^2} f(T/\mu)$$

implies $\theta_{,0} = 0$ , $T = 0$ and $S = 0$ .

Hence, the main point is that, assuming a barotropic equation of state $p=p(\rho)$, the dark energy temperature and entropy are zero.

# Conclusions

- Using the TF correspondence in the cosmological context most of the DE models can be represented by a kinetic k-essence
- A consistent grandcanonical description of DE involves two variables: the temperature *T* and the chemical potential *μ*
- The resulting thermodynamic equations do not require a negative entropy even in the phantom case, i.e., when NEC is violated
- Using the grandcanonical partition function in a saddle point approximation a barotropic equation of state yields $\mu \neq 0$, but $T=0$, $S=0$.