Tridiagonal Algebra and Exact Solvability

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Outline

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The asymmetric simple exclusion process(ASEP)

has become a paradigm in nonequilibrium physics due to its simplicity, rich behaviour and wide range of applicability.

It is an exacltly solvable model of an open many-particle stochastic system interacting with hard core exclusion.

Introduced originally as a simplified model of one dimensional transport for phenomena like

hopping conductivity and kinetics of biopolimerization,

it has found applications from traffic flow, to interface growth, shock formation, hydrodynamic systems obeying the noisy Burger equation, problems of sequence alignment in biology.

At large time the ASEP exibits relaxation

to a steady state,

and even after the relaxation it has

a nonvanishing current.

An intriguing feature is the occurrence of

boundary induced phase transitions

and the fact that

the stationary bulk properties depend strongly

on the boundary rates.

The ASEP is a stochastic process described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2...., n-1$ at a site $i = 1, 2, ...L$ of a linear chain. A state on the lattice at a time t is determined by the occupation numbers s_i and a transition to another configuration s_i' during an infinitesimal time step dt is given by the probability $\Gamma(s, s')dt$. The rates $\Gamma \equiv \Gamma_{jl}^{ik}$ are assumed to be independent from the position in the bulk. At the boundaries, i.e. sites 1 and L additional processes can take place with rates L and R. Due to probability conservation

$$\Gamma(s, s) = -\sum_{s' \neq s} \Gamma(s', s) \tag{1}$$

DIFFUSION - $\Gamma_{ki}^{ik} = g_{ik}$

Processes with exclusion - a site can be either empty or occupied by a particle of a given type.

In the set of occupation numbers $(s_1, s_2, ..., s_L)$ specifying a configuration of the system

$s_i = 0$ if a site i is empty,

$s_i = 1$ if there is a first-type particle at a site $i, \ldots,$

$s_i = n - 1$ if there is an $(n-1)$th-type particle at a site i.

- $g_{ik}dt$ - $i, k = 0, 1, 2, ..., n-1$ - with $i < k$,

g_{ik} are the probability rates of hopping to the left,

g_{ki} - to the right.

The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle, or each of the sites is occupied by a particle of a different type.

The n-species SYMMETRIC simple exclusion process - lattice gas model of particle hopping with a constant rate $g_{ik} = g_{ki} = g$.

The n-species ASYMMETRIC simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas.

The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping.

-The number of particles in the bulk is conserved and this is the case of periodic boundary conditions.

-In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density and additional processes can take place at the boundaries.

The master equation for the time evolution of a stochastic system

$$\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s,s')P(s',t) \qquad (2)$$

is mapped to a Schroedinger equation for a quantum Hamiltonian in imaginary time

$$\frac{dP(t)}{dt} = -HP(t) \qquad (3)$$

where

$$H = \sum_{j} H_{j,j+1} + H^{(L)} + B^{(R)} \qquad (4)$$

The ground state of this in general non-hermitean Hamiltonian corresponds to the stationary probability distribution of the stochastic dynamics.

The mapping provides a connection with integrable quantum spin chains.

Example: A relation to the integrable spin 1/2 XXZ quantum spin chain Hamiltonian H_{XXZ} with anisotropy $\Delta = \frac{(q+q^{-1})}{2}$ and most general non diagonal boundary terms H^L and H^R through the similarity transformation $\Gamma = -qU_\mu^{-1} H_{XXZ} U_\mu$

MATRIX PRODUCT STATES APPROACH

The stationary probability distribution, i.e. the ground state of the quantum Hamiltonian is expressed as a product of (or a trace over) matrices that form representation of a quadratic algebra determined by the dynamics of the process. (Derrida et al. - ASEP with open boundaries; 3-species diffusion-type, reaction-diffusion processes)

ANZATZ

Any zero energy eigenstate of a Hamiltonian with nearest neighbour interaction in the bulk and single site boundary terms can be written as a matrix product state with respect to a quadratic algebra

$$\Gamma_{jl}^{ik} D_i D_k = x_l D_j - x_j D_l$$

DIFFUSION - $\Gamma_{ki}^{ik} = g_{ik}$

DIFFUSION ALGEBRA

$$g_{ik}D_iD_k - g_{ki}D_kD_i = x_kD_i - x_iD_k \qquad (5)$$

where $i, k = 0, 1, ...n-1$ and x_i satisfy

$$\sum_{i=0}^{n-1} x_i = 0$$

This is an algebra with INVOLUTION, hence hermitean D_i

$$D_i = D_i^{+}, \qquad g_{ik}^{+} = g_{ki} \qquad x_i = -x_i^{+} \qquad (6)$$

(or $D_i = -D_i^{+}$, if $g_{ik} = g_{ki}^{+}$).

PROBABILITY DISTRIBUTION:

- periodic boundary conditions

$$P(s_1, ...s_L) = Tr(D_{s_1}D_{s_2}...D_{s_L}) \qquad (7)$$

-open systems with boundary processes

$$P(s_1, ...s_L) = < w|D_{s_1}D_{s_2}...D_{s_L}|v > \qquad (8)$$

the vectors $|v>$ and $<w|$ are defined by

$$< w|(L_i^kD_k + x_i) = 0, \qquad (R_i^kD_k - x_i)|v >= 0 \qquad (9)$$

where at site 1 (left) and at site L (right) the particle i is replaced by the particle k with probabilities $L^i_k dt$ and $R^i_k dt$ respectively.

$$L^i_i = -\sum_{j=0}^{L-1} L^i_j, \qquad R^i_i = -\sum_{j=0}^{L-1} R^i_j \tag{10}$$

THUS to find the stationary probability distribution one has to compute traces or matrix elements with respect to the vectors $|v>$ and $<w|$ of monomials of the form

$$D^{m_1}_{s_1} D^{m_2}_{s_2} \ldots.. D^{m_L}_{s_L} \tag{11}$$

The problem to be solved is twofold - Find a representation of the matrices D that is a solution of the quadratic algebra and match the algebraic solution with the boundary conditions.

The advantage of the matrix product state method

is that important physical properties and quantities

like multiparticle correlaton functions, currents,

density profiles, phase diagrams can be obtained

once the representations of the matrix quadratic algebra

and the boundary vectors are known.

EXACT SOLVABILITY of the ASYMMETRIC EXCLUSION MODEL
OPEN DIFFUSION SYSTEM COUPLED at the BOUNDARIES to EXTERNAL RESERVOIRS

- configuration set $s_1, s_2, ..., s_L$ where $s_i = 0$ if a site $i = 1, 2, ..., L$ is empty and $s_i = 1$ if a site i is occupied by a particle

- particles hop with a bulk probability $g_{01}dt$ to the left and with a probability $g_{10}dt$ to the right

- at the left boundary a particle can be added with probability αdt and removed with probability γdt

- at the right boundary it can be removed with probability βdt and added with probability δdt

right probability rate $g_{01} = q$
left probability rate $g_{10} = 1$

- quadratic algebra $D_1 D_0 - q D_0 D_1 = x_1 D_0 - x_0 D_1$

- boundary conditions: $(x_0 = -x_1 = 1)$

$$\begin{aligned} (\beta D_1 - \delta D_0)|v\rangle &= |v\rangle \\ \langle w|(\alpha D_0 - \gamma D_1) &= \langle w|. \end{aligned} \tag{12}$$

For a given configuration $(s_1, s_2, ..., s_L)$
the stationary probability is given by

$$P(s) = \frac{\langle w|D_{s_1} D_{s_2} ... D_{s_L}|v\rangle}{Z_L}, \tag{13}$$

$D_{s_i} = D_1$ if a site $i = 1, 2, ..., L$ is occupied
$D_{s_i} = D_0$ if a site i is empty and

$$Z_L = \langle w|(D_0 + D_1)^L|v\rangle$$

is the normalization factor to the stationary probability distribution.

Within the matrix-product ansatz, one can evaluate physical quantities such as:

- the mean density $\langle s_i \rangle$ at a site i

$$\langle s_i \rangle = \frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{L-i}|v\rangle}{Z_L}$$

- the current J through a bond between site i and site $i+1$,

$$J = \langle s_i(1 - s_{i+1}) - q(1 - s_i)s_{i+1}\rangle$$

$$= \frac{\langle w|(D_0 + D_1)^{i-1}(D_1 D_0 - q D_0 D_1)(D_0 + D_1)^{L-i-1}|v\rangle}{Z_L}$$

hence

$$J = \frac{Z_{L-1}}{Z_L}$$

- the two-point correlation function $\langle s_i s_j \rangle$

$$\frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{j-i-1} D_1 (D_0 + D_1)^{L-j}|v\rangle}{Z_L}$$

- higher correlation functions.

BOUNDARY ASKEY - WILSON ALGEBRA
of the ASYMMETRIC EXCLUSION PROCESS
with incoming and outgoing particles at the left and right boundaries

4 boundary parameters $\alpha, \beta, \gamma, \delta$

and bulk parameter $0 < q < 1$

Hence 2 algebraic relations for the

operators $\alpha D_0, \beta D_1, \gamma D_1, \delta D_0$

$$\beta D_1 \alpha D_0 - q \alpha D_0 \beta D_1 = x_1 \beta \alpha D_0 - \alpha \beta D_1 x_0 \quad (14)$$

$$\gamma D_1 \delta D_0 - q \delta D_0 \gamma D_1 = x_1 \gamma \delta D_0 - \delta \gamma D_1 x_0 \quad (15)$$

or instead (for the second relation)

$$\delta D_0 \gamma D_1 - q^{-1} \gamma D_1 \delta D_0 = q^{-1} x_0 \delta \gamma D_1 - q^{-1} \gamma \delta D_0 x_1 \quad (16)$$

To form two linearly independant boundary operators

$$B^R = \beta D_1 - \delta D_0, \qquad B^L = -\gamma D_1 + \alpha D_0$$

we use the $U_q(sl(2))$ algebra in the form of a deformed (u, v) algebra (to include all applications of the MPA quadratic algebra)

Special cases:
$U_q(su(2))$ $((u, -u),\ u < 0)$, a particular q-oscilator algebra $cu_q(2)$ $((u, u),\ u > 0)$ and two isomorphic ones $eu_q^{\pm}(2)$ $(uv = 0)$.

Defining commutation relations:

$$[N, A_{\pm}] = \pm A_{\pm} \qquad [A_-, A_+] = uq^N + vq^{-N} \tag{17}$$

Central element

$$Q = A_+ A_- + \frac{vq^N - uq^{1-N}}{1-q} \tag{18}$$

Representations in a basis $|n, \kappa\rangle$

a positive discrete series D_κ^+ defined by

$N|n, \kappa\rangle = (\kappa + n)|n, \kappa\rangle$, $A_-|n, \kappa\rangle = r_n|n-1, \kappa\rangle$,
$A_+|n, \kappa\rangle = r_{n+1}|n+1, \kappa\rangle$,

$$r_n^2 = \frac{(1-q^n)(vq^\kappa + uq^{1-n-\kappa})}{1-q}$$

$|0, \kappa\rangle$ is the vacuum with $r_0 = 0$.

The representation is infinite-dimensional if for all n

$$vq^\kappa + uq^{1-n-\kappa} > 0$$

fulfilled for $U_q(sl_2)$ $(\kappa > 0)$,

and finite-dimensional of dimension $l+1$ in the $U_q(su_2)$ case, if for some $n = l$

$$-uq^\kappa + uq^{-l-\kappa} = 0 \tag{19}$$

REPRESENTATION of the BOUNDARY OPERATORS

$$\beta D_1 - \delta D_0 =$$
$$-\frac{x_1\beta}{\sqrt{1-q}} q^{N/2} A_+ - \frac{x_0\delta}{\sqrt{1-q}} A_- q^{N/2}$$
$$-\frac{x_1\beta q^{1/2} + x_0\delta}{1-q} q^N - \frac{x_1\beta + x_0\delta}{1-q}$$

$$\alpha D_0 - \gamma D_1 =$$
$$\frac{x_0\alpha}{\sqrt{1-q}} q^{-N/2} A_+ + \frac{x_1\gamma}{\sqrt{1-q}} A_- q^{-N/2}$$
$$+\frac{x_0\alpha q^{-1/2} + x_1\gamma}{1-q} q^{-N} + \frac{x_0\alpha + x_1\gamma}{1-q} \quad (20)$$

SEPARATE the SHIFT PARTS and DENOTE the REST by A and A^*

$$\begin{aligned} \beta D_1 - \delta D_0 &= A - \frac{x_1\beta + x_0\delta}{1-q} \\ \alpha D_0 - \gamma D_1 &= A^* + \frac{x_0\alpha + x_1\gamma}{1-q} \end{aligned} \quad (21)$$

HENCE the OPERATORS A and A^*

$$A = \beta D_1 - \delta D_0 + \frac{x_1\beta + x_0\delta}{1-q} \qquad (22)$$
$$A^* = \alpha D_0 - \gamma D_1 - \frac{x_0\alpha + x_1\gamma}{1-q}$$

and their [q-COMMUTATOR]

$$[A, A^*]_q = q^{1/2} A A^* - q^{-1/2} A^* A \qquad (23)$$

form a closed linear algebra - the ASKEY-WILSON ALGEBRA

$$[[A, A^*]_q, A]_q = -\rho A^* - \omega A - \eta \qquad (24)$$
$$[A^*, [A, A^*]_q]_q = -\rho^* A - \omega A^* - \eta^*$$

with REPRESENTATION-DEPENDENT STRUCTURE CONSTANTS

$$
\begin{aligned}
-\rho &= x_0 x_1 \beta\delta q^{-1}(q^{1/2}+q^{-1/2})^2, \\
-\rho^* &= x_0 x_1 \alpha\gamma q^{-1}(q^{1/2}+q^{-1/2})^2
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
-\omega &= (x_1\beta + x_0\delta)(x_1\gamma + x_0\alpha) \\
&- (x_1^2\beta\gamma + x_0^2\alpha\delta)(q^{1/2}-q^{-1/2})Q
\end{aligned} \tag{26}
$$

$$
\eta = q^{1/2}(q^{1/2}+q^{-1/2})\times
$$
$$
\left(x_0 x_1 \beta\delta(x_1\gamma + x_0\alpha)Q - \frac{(x_1\beta + x_0\delta)(x_1^2\beta\gamma + x_0^2\alpha\delta)}{q^{1/2}-q^{-1/2}}\right)
$$

$$
\eta^* = q^{1/2}(q^{1/2}+q^{-1/2})\times
$$
$$
\left(x_0 x_1 \alpha\gamma(x_1\beta + x_0\delta)Q + \frac{(x_0\alpha + x_1\gamma)(x_0^2\alpha\delta + x_1^2\beta\gamma)}{q^{1/2}-q^{-1/2}}\right)
$$

AW algebra first considered by A. Zhedanov, recently discussed in a more general framework of a tridiagonal algebra (Terwilliger) associative algebra (with a unit) generated by a tridiagonal pair of operators A, A^* and defining relations

$$[A,[A[A,A^*]_q]_{q^{-1}} - \gamma(AA^* + A^*A)] = \rho[A,A^*] \tag{27}$$

$$[A^*,[A^*[A^*,A]_q]_{q^{-1}} - \gamma^*(AA^* + A^*A) = \rho^*[A^*,A] \tag{28}$$

In the general case a tridiagonal pair is determined by the sequence of scalars $\beta, \gamma, \gamma^*, \rho, \rho^*$ from a field K. Tridiagonal pairs have been classified according to the dependence on the scalars.

Affine transformations act on tridiagonal pairs

$$A \to tA + c, \qquad A^* \to t^*A^* + c^* \tag{29}$$

with t, t^*, c, c^* some scalars

can be used to bring a tridiagonal pair in a reduced form with $\gamma = \gamma^* = 0$.

Important Examples:

the q-Serre relations

$\beta = q + q^{-1} \qquad \gamma = \gamma^* = \rho = \rho^* = 0$

$$[A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2] = 0 \quad (30)$$
$$[A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2}] = 0$$

the Dolan-Grady relations with

$\beta = 2, \gamma = \gamma^* = 0, \rho = k^2, \rho^* = k^{*2}$

$$[A, [A, [A, A^*]]] = k^2[A, A^*] \quad (31)$$
$$[A^*, [A^*, [A^*, A]]] = k^{*2}[A^*, A]$$

The AW algebra possesses important properties

that allow to obtain its ladder representations, spectra, overlap functions.

Namely, there exists a basis (of orthogonal polynomials) f_r

according to which the operator A is diagonal and the operator A^* is tridiagonal.

There exists a dual basis f_p in which the operator A^* is diagonal and the operator A is tridiagonal.

The overlap function of the two basis $\langle s|r\rangle = \langle f_s^*|f_r\rangle$ is expressed in terms of the Askey-Wilson polynomials.

Relation of the BOUNDARY ALGEBRA to the BASIC REPRESENTATION of the AW ALGEBRA

1. Divide the boundary eqs. by β and α,

$$\begin{aligned} B^R &= \beta D_1 - \delta D_0 \to D_1 - \frac{\delta}{\beta} D_0 \\ B^L &= -\gamma D_1 + \alpha D_0 \to D_0 - \frac{\gamma}{\alpha} D_1 \end{aligned} \qquad (32)$$

2. Hence a new sequence of scalars for the TD pair

$$\rho/\beta, \quad \rho^*/\alpha, \quad \omega/\alpha\beta, \quad \eta/\alpha\beta, \quad \eta^*/\alpha\beta$$

3. Set $x_0 = -x_1 = s$ where s is a free parameter from $x_0 + x_1 = 0$.

4. Rescale the generators $A \equiv \frac{1}{\beta} A$ and $A^* \equiv \frac{1}{\alpha} A^*$

$$\begin{aligned} A &\to (q^{-1/2} - q^{1/2}) \frac{1}{q^{-1/2} s \sqrt{bd}} A \\ A^* &\to (q^{-1/2} - q^{1/2}) \frac{\sqrt{bd}}{s} A^* \end{aligned} \qquad (33)$$

The tridiagonal relations for the transformed operators read

$$\begin{aligned} [A,[A[A,A^*]_q]_{q^{-1}} &= -(q-q^{-1})^2[A,A^*] \quad (34) \\ [A*,[A*[A*,A]_q]_{q^{-1}} &= -abcdq^{-1}q-q^{-1})^2[A*,A] \end{aligned}$$

where $abcd = \frac{\gamma}{\alpha}\frac{\delta}{\beta}$.

Let $p_n = p_n(x;a,b,c,d)$ denote the nth Askey-Wilson polynomial depending on four parameters a,b,c,d

$$p_n =_4 \Phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{matrix} |q;q \right) \quad (35)$$

with $p_0 = 1$, $x = y + y^{-1}$ and $0 < q < 1$.

The basic representation π is in the space of symmetric Laurent polynomials $f[y]$ with a basis $(p_0, p_1, ...)$

$$Af[y] = (y+y^{-1})f[y], \quad A^*f[y] = \mathcal{D}f[y] \quad (36)$$

where $\mathcal{D}$ is the second order q-difference operator having the Askey-Wilson polynomials p_n as eigenfunctions, namely a linear transformation given by

$$\mathcal{D}f[y] = (1 + abcdq^{-1})f[y]$$
$$+ \frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)}(f[qy] - f[y])$$
$$+ \frac{(a-y)(b-y)(c-y)(d-y)}{(1-y^2)(q-y^2)}(f[q^{-1}y] - f[y])$$

with $\mathcal{D}(1) = 1 + abcdq^{-1}$. The eigenvalue equation for the joint eigenfunctions p_n reads

$$\mathcal{D}p_n = \lambda_n^* p_n, \qquad \lambda_n^* = q^{-n} + abcdq^{n-1} \tag{37}$$

and the operator A^* is represented by an infinite-dimensional matrix diag$(\lambda_0^*, \lambda_1^*, \lambda_2^*, ...)$. The operator $Ap_n = xp_n$ is represented by a tridiagonal matrix

$$\mathcal{A} = \begin{pmatrix} a_0 & c_1 & & \\ b_0 & a_1 & c_2 & \\ & b_1 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix} \tag{38}$$

whose matrix elements are obtained from the three term recurrence relation for the Askey-Wilson polynomials

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0 \tag{39}$$

The explicit form of the matrix elements of A reads

$$a_n = a + a^{-1} - b_n - c_n \tag{40}$$

$$b_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})} \tag{41}$$

$$c_n = \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})} \tag{42}$$

The basis is orthogonal with the orthogonality condition for the Askey-Wilson polynomials

$P_n(x;a,b,c,d|q) = a^{-n}(ab,ac,ad;q)_n p_n$

$$\int_{-1}^{1} \frac{w(x)}{2\pi\sqrt{1-x^2}} P_m P_n dx = h_n \delta_{mn} \qquad (43)$$

where $w(x) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)}$,

$h(x,\mu) = \prod_{k=0}^{\infty}[1 - 2\mu x q^k + \mu^2 q^{2k}]$,

and

$$h_n = \frac{(abcdq^{n-1};q)_n (abcdq^{2n};q)_\infty}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty} \qquad (44)$$

Result: A representation π with basis $(p_0, p_1, p_2, ...)^t$

$\pi(D_1 - \frac{\delta}{\beta}D_0)$ is diagonal with eigenvalues

$$\lambda_n = \frac{1}{1-q}\left(bq^{-n} + dq^{n-1}\right) + \frac{1}{1-q}(1+bd) \quad (45)$$

and $\pi(D_0 - \frac{\gamma}{\alpha}D_1)$ is tridiagonal

$$\pi(D_0 - \frac{\gamma}{\alpha}D_1) = \frac{1}{1-q}b\mathcal{A}^t + \frac{1}{1-q}(1+ac) \quad (46)$$

The dual representation π^* has a basis $p_0, p_1, p_2, ...$

with $\pi^*(D_0 - \frac{\gamma}{\alpha}D_1)$ diagonal with eigenvalues

$$\lambda_n^* = \frac{1}{1-q}\left(aq^{-n} + cq^{n}\right) + \frac{1}{1-q}(1+ac) \quad (47)$$

and $\pi^*(D_1 - \frac{\delta}{\beta}D_0)$ tridiagonal

$$\pi^*(D_1 - \frac{\delta}{\beta}D_0) = \frac{1}{1-q}a\mathcal{A} + \frac{1}{1-q}(1+bd) \quad (48)$$

The choice
$\langle w| = h_0^{-1/2}(p_0, 0, 0, ...)$, $|v\rangle = h_0^{-1/2}(p_0, 0, 0, ...)^t$
(h_0 is a normalization)

as eigenvectors of the diagonal matrices

$\pi(D_1 - \frac{\delta}{\beta}D_0)$ and $\pi^*(D_0 - \frac{\gamma}{\alpha}D_1)$

yields a solution to the boundary equations which uniquely relate a, b, c, d to $\alpha, \beta, \gamma, \delta$.

Namely
$a = \kappa^*_+(\alpha, \gamma)$, $b = \kappa_+(\beta, \delta)$,
$c = \kappa^*_-(\alpha, \gamma)$, $d = \kappa_-(\beta, \delta)$

where $\kappa^{(*)}_\pm(\nu, \tau)$ ($\equiv \kappa^{(*)}_\pm$) is

$$\kappa^{(*)}_\pm = \frac{-(\nu - \tau - (1-q)) \pm \sqrt{(\nu - \tau - (1-q))^2 + 4\nu\tau}}{2\nu} \quad (49)$$

EACH BOUNDARY OPERATOR and the TRANSFER MATRIX $D_0 + D_1$

form an ISOMORPHIC TRIDIAGONAL PAIR

HENCE $$(D_0 + D_1)p_n = (2 + x)p_n$$

and using the orthogonality relation in the form

$$1 = h_0^{-1} \int dy w(y + y^{-1})|p(y + y^{-1})\rangle\langle p(y + y^{-1})|$$

one obtains (omitting the long technical datails)

A. $a > 1, \quad a > b$

$$Z_L^a \simeq \left(\frac{(1+a)(1+a^{-1})}{1-q}\right)^L$$

$$J \simeq (1-q)\frac{a}{(1+a)^2}$$

B. $b > 1, \quad b > a$

$$Z_L^b \simeq \left(\frac{(1+b)(1+b^{-1})}{1-q}\right)^L$$

$$J \simeq (1-q)\frac{b}{(1+b)^2}$$

and analogously, the correlation functions, the density profile, etc.

CONCLUSION

BOUNDARY ASKEY-WILSON ALGEBRA OF THE OPEN ASEP IS

THE LINEAR COVARIANCE ALGEBRA OF THE BULK $U_q(su(2))$ SYMMETRY

AND ALLOWS FOR THE EXACT SOLVABILITY.