Action-angle variables for geodesic motion on the resolved metric cone over Sasaki-Einstein space $T^{1,1}$

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Contact geometry (1)

A $(2n+1)$-dimensional manifold M is a *contact manifold* if there exists a 1-form η (called a contact 1-form) on M such that

$$\eta \wedge (d\eta)^{n-1} \neq 0 \,.$$

Associated with a contact form η there exists a unique vector field R_η called the *Reeb vector field* defined by the contractions (interior products):

$$\boldsymbol{i}(R_\eta)\eta = 1 \,,$$

$$\boldsymbol{i}(R_\eta)d\eta = 0 \,.$$

Contact geometry (2)

Simple example (1)

On $\mathbb{R}^3$ with cartesian coordinates (x, y, z)

Contact form

$$\eta = dz + xdy$$

Reeb vector

$$R_\eta = \frac{\partial}{\partial z}$$

Riemannian metric

$$\begin{aligned} g &= (dx)^2 + (dy)^2 + \eta^2 \\ &= (dx)^2 + (1 + x^2)(dy)^2 + 2xdy\,dz + (dz)^2 \end{aligned}$$

Contact plane $\mathit{ker}\,\eta$ at a point (x, y, z) is spanned by the vectors

$$\begin{aligned} X_1 &= \partial_x \\ X_2 &= x\partial_z - \partial_y \end{aligned}$$

Planes appear to twist along the x axis (“propeller picture”)

Contact geometry (3)

Simple example (2)

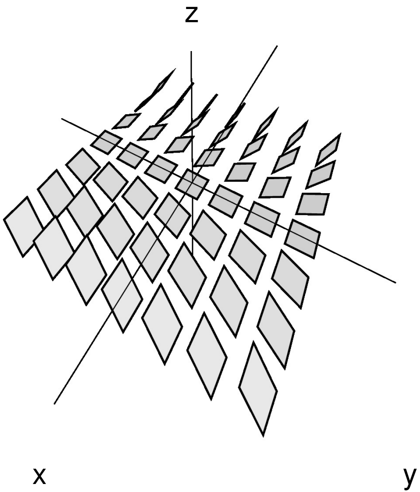

Figure : The contact structure $ker\,(dz + x\,dy)$.

Sasakian geometry (1)

A simple and direct definition of the Sasakian structures is the following:

A compact Riemannian manifold (M, g) is **Sasakian** if and only if its metric cone $(C(M) \cong \mathbb{R}_+ \times M\,, \, \bar{g} = dr^2 + r^2\, g)$ is Kähler.

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. The Sasakian manifold (M, g) is naturally isometrically embedded into the metric cone via the inclusion $M = \{r = 1\} = \{1\} \times M \subset C(M)$.

Kähler form on metric cone

$$\omega = \frac{1}{2} d(r^2 \eta) = r dr \wedge \eta + \frac{1}{2} r^2 d\eta\,.$$

Sasakian geometry (2)

Sasaki-Einstein geometry is naturally “sandwiched” between two Kähler-Einstein geometries as shown in the following proposition:

Let (M, g) be a Sasaki manifold of dimension $2n - 1$. Then the following are equivalent

(1) (M, g) is Sasaki-Einstein with $Ric_g = 2(n-1)g$;

(2) The Kähler cone $(C(M), \bar{g})$ is Ricci-flat, i.e. Calabi-Yau $Ric_{\bar{g}} = 0$;

(3) The transverse Kähler structure to the Reeb foliation $\mathcal{F}_{R_\eta}$ is Kähler-Einstein with $Ric_T = 2ng_T$.

Complete integrability on $T^{1,1}$ space (1)

One of the most familiar example of homogeneous toric Sasaki-Einstein five-dimensional manifold is the space $T^{1,1} = S^2 \times S^3$ endowed with the following metric

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \,.$$

The global defined contact 1-form is

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \,.$$

The Reeb vector field R_η has the form

$$R_\eta = 3\frac{\partial}{\partial\psi} \,,$$

and it is easy to see that $\eta(R_\eta) = 1$.

Complete integrability on $T^{1,1}$ space (2)

The Hamiltonian describing the geodesic flow is

$$H = \frac{1}{2} g^{ij} p_i p_j \,,$$

where g^{ij} is the inverse metric of $T^{1,1}$ space and $p_i = g_{ij}\,\dot{x}^j$ are the conjugate momenta to the coordinates $(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$

$$\begin{aligned}
p_{\theta_i} =& \frac{1}{6}\dot{\theta}_i \quad , \quad i = 1, 2 \\
p_{\phi_1} =& \frac{1}{6}\sin^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos\theta_1\,\dot{\psi} + \\
& \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot{\phi}_2 \,, \\
p_{\phi_2} =& \frac{1}{6}\sin^2\theta_2\,\dot{\phi}_2 + \frac{1}{9}\cos^2\theta_2\,\dot{\phi}_2 \\
& + \frac{1}{9}\cos\theta_2\,\dot{\psi} + \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot{\phi}_1 \,, \\
p_{\psi} =& \frac{1}{9}\,\dot{\psi} + \frac{1}{9}\cos\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos\theta_2\,\dot{\phi}_2 \,,
\end{aligned}$$

Complete integrability on $T^{1,1}$ space (3)

The conserved Hamiltonian takes the form:

$$H = 3\left[p_{\theta_1}^2 + p_{\theta_2}^2 + \frac{1}{\sin^2\theta_1}(p_{\phi_1} - \cos\theta_1 p_\psi)^2 + \frac{1}{\sin^2\theta_2}(p_{\phi_2} - \cos\theta_2 p_\psi)^2\right] + \frac{9}{2}p_\psi^2 \,.$$

Taking into account the isometries of $T^{1,1}$, the momenta p_{ϕ_1}, p_{ϕ_2} and p_ψ are conserved.

Complete integrability on $T^{1,1}$ space (4)

Two total $SU(2)$ angular momenta are also conserved:

$$\vec{j}_1^{\,2} = p_{\theta_1}^2 + \frac{1}{\sin^2\theta_1}(p_{\phi_1} - \cos\theta_1 p_\psi)^2 + p_\psi^2 \,,$$
$$\vec{j}_2^{\,2} = p_{\theta_2}^2 + \frac{1}{\sin^2\theta_2}(p_{\phi_2} - \cos\theta_2 p_\psi)^2 + p_\psi^2 \,.$$

Complete integrability on metric cone (1)

Conifold metric is

$$ds_{mc}^2 = dr^2 + \frac{r^2}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) + \frac{r^2}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 .$$

On the metric cone the geodesic flow is described by

$$H_{C(T^{1,1})} = \frac{1}{2}\,p_r^2 + \frac{1}{r^2}\,\tilde{H}\,,$$

where the radial momentum is

$$p_r = \dot{r}\,.$$

Hamiltonian $\tilde{H}$ has a similar structure as in $T^{1,1}$, but constructed with momenta $\tilde{p}_i$ related to momenta p_i by

$$\tilde{p}_i = r^2 g_{ij}\,\dot{x}^j = r^2\,p_i\,.$$

Complete integrability on metric cone (2)

Radial dynamics is independent of the dynamics of the base manifold $T^{1,1}$ and $\tilde{H}$ is a constant of motion. The Hamilton equations of motion for $\tilde{H}$ on $T^{1,1}$ have the standard form in terms of a new time variable $\tilde{t}$ given by

$$\frac{dt}{d\tilde{t}} = r^2 \,.$$

Concerning the constant of motions, they are the conjugate momenta $(\tilde{p}_{\phi_1}, \tilde{p}_{\phi_2}, \tilde{p}_{\psi})$ associated with the cyclic coordinates (ϕ_1, ϕ_2, ψ) and two total $SU(2)$ momenta

$$\tilde{\mathbf{j}}_i^2 = r^4 \, \mathbf{j}_i^2 \quad , \quad i = 1, 2 \,.$$

Together with the Hamiltonian $H_{C(T^{1,1})}$, they ensure the complete integrability of the geodesic flow on the metric cone.

Complete integrability on metric cone (3)

Considering a particular level set E of $H_{C(T^{1,1})}$, we get for the radial motion

$$p_r^2 = \dot{r}^2 = 2E - \frac{2}{r^2}\,\tilde{H}\,.$$

The turning point of the radial motion is determined by

$$\dot{r} = 0 \quad \Longrightarrow \quad r_* = \sqrt{\frac{\tilde{H}}{E}}\,.$$

Projecting the geodesic curves onto the base manifold $T^{1,1}$ we can evaluate the total distance transversed in the Sasaki-Einstein space between the limiting points as $t \to -\infty$ and $t \to +\infty$

$$d = \sqrt{2\tilde{H}} \int_{-\infty}^{\infty} \frac{dt}{r_*^2 + 2E\,t^2} = \pi\,.$$

Complete integrability on metric cone (4)

Note: Radial motion is unbounded and consequently Hamiltonian $H_{C(T^{1,1})}$ does not admit a formulation in terms of action-angle variables.

Key idea: split the mechanical system into a “radial” and an “angular” part. The angular part is a compact subsystem spanned by the set of variables $q = (\theta_1, \theta_2, \phi_1, \phi_2, \psi)$ which can be formulated in terms of action-angle variables $(I_i, \Phi_i^0)\,,\; i = \theta_1, \theta_2, \phi_1, \phi_2, \psi$. By adding the radial part $r \in (0, \infty)$, Hamiltonian $H_{C(T^{1,1})}$ can be put in the form

$$H_{C(T^{1,1})} = \frac{1}{2}\, p_r^2 + \frac{1}{r^2}\, \tilde{H}(I_i)\,.$$

Concerning the action-angle variables (I_i, Φ_i^0) corresponding to the compact angular subsystem, they can be determined by a standard technique.

Complete integrability on metric cone (5)

Let us consider a particular level set E of the energy. Using complete separability, Hamilton's principal function can be written in the form:

$$
\begin{aligned}
S(r,q,\alpha,t) = \ & S_0(r,q,\alpha) - Et = S_r(r,\alpha) + \tilde{S}_0(q,\alpha) - Et \\
= & \ S_r(r,\alpha) + \sum_{j=1,2} S_{\theta_j}(\theta_j,\alpha) + \sum_{j=1,2} S_{\phi_j}(\phi_j,\alpha) + S_\psi(\psi,\alpha) - Et
\end{aligned}
$$

where α is a set of constants of integration.
Hamilton-Jacobi equation is

$$
\begin{aligned}
E = \ & \frac{1}{2}\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{9}{2r^2}\left(\frac{\partial S_\psi}{\partial \psi}\right)^2 \\
& + \frac{3}{r^2}\sum_{i=1,2}\left\{\left(\frac{\partial S_{\theta_i}}{\partial \theta_i}\right)^2 + \frac{1}{\sin^2\theta_i}\left[\left(\frac{\partial S_{\phi_i}}{\partial \phi_i}\right) - \cos\theta_i\left(\frac{\partial S_\psi}{\partial \psi}\right)\right]^2\right\}
\end{aligned}
$$

Complete integrability on metric cone (6)

Since the variables (ϕ_1, ϕ_2, ψ) are cyclic, we have

$$S_{\phi_1} = \tilde{p}_{\phi_1} \cdot \phi_1 = \alpha_{\phi_1} \cdot \phi_1 \,,$$
$$S_{\phi_2} = \tilde{p}_{\phi_2} \cdot \phi_2 = \alpha_{\phi_2} \cdot \phi_2 \,,$$
$$S_{\psi} = \tilde{p}_{\psi} \cdot \psi = \alpha_{\psi} \cdot \psi \,,$$

where $\alpha_{\phi_1}, \alpha_{\phi_2}, \alpha_{\psi}$ are constants of integration.
The corresponding action variables are:

$$I_{\phi_1} = \frac{1}{2\pi} \oint \frac{\partial S_{\phi_1}}{\partial \phi_1} d\phi_1 = \alpha_{\phi_1} \,,$$
$$I_{\phi_2} = \frac{1}{2\pi} \oint \frac{\partial S_{\phi_1 2}}{\partial \phi_2} d\phi_2 = \alpha_{\phi_2} \,,$$
$$I_{\psi} = \frac{1}{4\pi} \oint \frac{\partial S_{\psi}}{\partial \psi} d\psi = \alpha_{\psi} \,.$$

Complete integrability on metric cone (7)

Next we deal with the coordinates $\theta_i\,,\ i=1,2$.
From Hamilton-Jacobi equation we get

$$\left(\frac{\partial S_{\theta_i}}{\partial\theta_i}\right)^2+\frac{1}{\sin^2\theta_i}\left(\alpha_{\phi_i}-\alpha_\psi\cos\theta_i\right)^2=\alpha_{\theta_i}^2\quad,\quad i=1,2\,,$$

where $\alpha_{\theta_i}\,,\ i=1,2$ are constants.
The corresponding action variables $I_{\theta_i}\,,\ i=1,2$, are

$$I_{\theta_i}=\frac{1}{2\pi}\oint\left(\alpha_{\theta_i}^2-\frac{(\alpha_{\phi_i}-\alpha_\psi\cos\theta_i)^2}{\sin^2\theta_i}\right)^{\frac{1}{2}}d\theta_i\quad,\quad i=1,2\,.$$

The limits of integrations are defined by the roots θ_{i-} and θ_{i+} of the expressions in the square root parenthesis and a complete cycle of θ_i involves going from θ_{i-} to θ_{i+} and back to θ_{i-} .
An efficient technique for evaluating I_{θ_i} is to extend θ_i to a complex variable z_i and interpret the integral as a closed contour integral in the z_i plane.

Complete integrability on metric cone (8)

At the end, we get

$$I_{\theta_i} = \left(\alpha_{\theta_i}^2 + \alpha_\psi^2\right)^{\frac{1}{2}} - \alpha_{\phi_i} \quad , \quad i = 1, 2\,.$$

We note that each S_i , $i = (\theta_1, \theta_2, \phi_1, \phi_2, \psi)$ depends on all action variables I_i. Then the angle variables are

$$\Phi_i^0 = \frac{\partial \tilde{S}_0}{\partial I_i} \quad , \quad i = (\theta_1, \theta_2, \phi_1, \phi_2, \psi)\,.$$

Their explicit expressions are quite involved and are not produced here.

Complete integrability on metric cone (9)

Finally, let us consider the radial part of the Hamilton's principal function.

From Hamilton-Jacobi equation have

$$\left(\frac{\partial S_r}{\partial r}\right)^2 = 2E - \frac{6}{r^2}(\alpha_{\theta_1}^2 + \alpha_{\theta_2}^2 + \frac{3}{2}\alpha_\psi^2)\,,$$

and

$$S_r(r,\alpha) = \int^r dr' \left(2E - \frac{6}{r'^2}(\alpha_{\theta_1}^2 + \alpha_{\theta_2}^2 + \frac{3}{2}\alpha_\psi^2)\right)^{\frac{1}{2}}\ .$$

Thus, the Hamilton's principal function is

$$S(E, I_i, r, \Phi_i^0) = \sqrt{2}\int^r dr' \sqrt{E - \frac{\tilde{H}(I_i)}{r'^2}} + \sum_i I_i \Phi_i^0\,,$$

where the sum extends over the action-angle variables $i = (\theta_1, \theta_2, \phi_1, \phi_2, \psi)$.

Complete integrability on metric cone (10)

The integral corresponding to the radial motion can be evaluated and, eventually, we get

$$S_r(r, I_i) = \sqrt{2E}\left(\sqrt{r^2 - r_*^2} - r_* \arctan\sqrt{\frac{r^2 - r_*^2}{r_*^2}}\right)$$

where the turning point r_* , in terms of action variables I_i , is

$$r_*^2 = \frac{3}{E}\left[(I_{\theta_1} + I_{\phi_1})^2 + (I_{\theta_1} + I_{\phi_1})^2 - \frac{1}{2}I_\psi^2\right] .$$

Complete integrability on resolved metric cone (1)

The metric cone associated with Sasaki-Einstein space $T^{1,1}$ is described by the following equation in four complex variables

$$\sum_{a=1}^{4} w_a^2 = 0\,.$$

Equation of the quadric can be rewritten using a matrix $\mathcal{W}$

$$\mathcal{W} = \frac{1}{\sqrt{2}}\,\sigma^a w_a = \frac{1}{\sqrt{2}} \begin{pmatrix} w_3 + i w_4 & w_1 - i w_2 \\ w_1 + i w_2 & -w_3 + i w_4 \end{pmatrix} \equiv \begin{pmatrix} X & U \\ V & Y \end{pmatrix}$$

where σ^a are the Pauli matrices for $a = 1, 2, 3$ and σ^4 is i times the unit matrix. The radial coordinate is defined by

$$r^2 = \mathrm{tr}\,(\mathcal{W}^\dagger \mathcal{W})\,.$$

In terms of the matrix $\mathcal{W}$, equation of quadric can be written as

$$\det \mathcal{W} = 0 \quad , \quad \text{i.e.} \quad XY - UV = 0\,.$$

Complete integrability on resolved metric cone (2)

Small resolution (1)

The small resolution is realized replacing $det\, W = 0$ by the pair of equations:

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0\,,$$

in which $(\lambda_1, \lambda_2) \in \mathbb{CP}^1$ are not both zero. Thus, in the region where $\lambda_1 \neq 0$, the pair (λ_1, λ_2) is uniquely characterized by the coordinate $\lambda = \lambda_2/\lambda_1$, while in the region where $\lambda_2 \neq 0$, the pair (λ_1, λ_2) is described by the coordinate $\mu = \lambda_1/\lambda_2$.
It turns out to be convenient to introduce a new radial coordinate

$$\rho^2 \equiv \frac{3}{2}\,\gamma\,,$$

where the function γ is given by the equation

$$\gamma^3 + 6\,a^2\,\gamma^2 - r^4 = 0\,,$$

with a the “resolution” parameter. It represents the radius of the sphere S^2 which replaces the point singularity at $r^2 = 0$.

Complete integrability on resolved metric cone (3)

Small resolution (2)

Metric of the resolved conifold (rc) can be written as

$$ds^2_{rc} = \kappa^{-1}(\rho)\, d\rho^2 + \frac{1}{9}\kappa(\rho)\rho^2(d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2)^2$$
$$+ \frac{1}{6}\rho^2(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) + \frac{1}{6}(\rho^2 + 6\,a^2)(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2)\,,$$

where

$$\kappa(\rho) \equiv \frac{\rho^2 + 9\,a^2}{\rho^2 + 6\,a^2}\,.$$

The resolved conifold metric is Ricci flat and has an explicit $SU(2) \times SU(2)$ invariant form. When the resolution parameter a goes to zero or when $\rho \to \infty$, the resolved conifold metric reduces to the standard conifold metric $g_{C(T^{1,1})}$. In fact the parameter a introduces an asymmetry between the two sphere.

Complete integrability on resolved metric cone (4)

Small resolution (3)

Conjugate momenta $(P_\rho, P_{\theta_1}, P_{\theta_2}, P_{\phi_1}, P_{\phi_2}, P_\psi)$ corresponding to the coordinates $(\rho, \theta_1, \theta_2, \phi_1, \phi_2, \psi)$ are:

$$
\begin{aligned}
P_\rho &= \kappa^{-1}(\rho)\dot{\rho} \\
P_{\theta_1} &= \frac{1}{6}\rho^2\dot{\theta}_1 \\
P_{\theta_2} &= \frac{1}{6}(\rho^2 + 6\,a^2)\dot{\theta}_2 \\
P_{\phi_1} &= \frac{1}{6}\rho^2 \sin^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\kappa(\rho)\rho^2\cos\theta_1(\cos\theta_1\,\dot{\phi}_1 + \cos\theta_2\,\dot{\phi}_2 + \dot{\psi}) \\
P_{\phi_2} &= \frac{1}{6}(\rho^2 + 6\,a^2)\sin^2\theta_2\,\dot{\phi}_2 \\
&\quad + \frac{1}{9}\kappa(\rho)\rho^2\cos\theta_2(\cos\theta_1\,\dot{\phi}_1 + \cos\theta_2\,\dot{\phi}_2 + \dot{\psi}) \\
P_\psi &= \frac{1}{9}\kappa(\rho)\rho^2(\cos\theta_1\,\dot{\phi}_1 + \cos\theta_2\,\dot{\phi}_2 + \dot{\psi})\,.
\end{aligned}
$$

Complete integrability on resolved metric cone (5)

Small resolution (4)

In terms of them, the Hamiltonian for the geodesic flow is

$$H_{rc} = \frac{\kappa(\rho)}{2} P_\rho^2 + \frac{9}{2}\,\frac{1}{\kappa(\rho)\,\rho^2} P_\psi^2 + \frac{3}{\rho^2}\,P_{\theta_1}^2 + \frac{3}{\rho^2 + 6\,a^2}\,P_{\theta_2}^2$$
$$+ \frac{3}{\rho^2 \sin^2\theta_1}\,(P_{\phi_1} - \cos\theta_1\,P_\psi)^2$$
$$+ \frac{3}{(\rho^2 + 6\,a^2)\,\sin^2\theta_2}\,(P_{\phi_2} - \cos\theta_2\,P_\psi)^2\,.$$

(ϕ_1, ϕ_2, ψ) are still cyclic coordinates and, accordingly, momenta $P_{\phi_1}, P_{\phi_2}, P_\psi$ are conserved. Taking into account the symmetry $SU(2) \times SU(2)$, total angular momenta

$$\mathbf{J}_1^2 = P_{\theta_1}^2 + \frac{1}{\sin^2\theta_1}(P_{\phi_1} - \cos\theta_1\,P_\psi)^2 + P_\psi^2 = \rho^4\,\mathbf{j}_1^2\,,$$
$$\mathbf{J}_2^2 = P_{\theta_2}^2 + \frac{1}{\sin^2\theta_2}(P_{\phi_2} - \cos\theta_2\,P_\psi)^2 + P_\psi^2 = (\rho^2 + 6a^2)^2\,\mathbf{j}_2^2\,,$$

are also conserved.

Complete integrability on resolved metric cone (6)

Small resolution (5)

The set of conserved quantities $(H_{rc}, P_{\phi_1}, P_{\phi_2}, P_{\psi}, \mathbf{J}_1^2, \mathbf{J}_2^2)$ ensures the complete integrability of geodesic motions on the resolved conifold. As it is expected, for $a = 0$ we recover the state of integrability on the standard metric cone of the Sasaki-Einstein space $T^{1,1}$.

Hamilton's principal function has the form:

$$S(\rho, q, \alpha, t) = S_\rho(\rho, \alpha) + \sum_{j=1,2} S_{\theta_j}(\theta_j, \alpha) + \sum_{j=1,2} S_{\phi_j}(\phi_j, \alpha) + S_\psi(\psi, \alpha) - Et\,.$$

Complete integrability on resolved metric cone (7)

Small resolution (6)

Hamilton-Jacobi equation becomes:

$$E = \frac{1}{2}\kappa(\rho)\left(\frac{\partial S_\rho}{\partial \rho}\right)^2 + \frac{3}{\rho^2}\left(\frac{\partial S_{\theta_1}}{\partial \theta_1}\right)^2 + \frac{3}{\rho^2 + 6a^2}\left(\frac{\partial S_{\theta_2}}{\partial \theta_2}\right)^2$$
$$+ \frac{3}{\rho^2 \sin^2\theta_1}\left[\left(\frac{\partial S_{\phi_1}}{\partial \phi_1}\right) - \cos\theta_1\left(\frac{\partial S_\psi}{\partial \psi}\right)\right]^2$$
$$+ \frac{3}{(\rho^2 + 6a^2)\sin^2\theta_2}\left[\left(\frac{\partial S_{\phi_2}}{\partial \phi_2}\right) - \cos\theta_2\left(\frac{\partial S_\psi}{\partial \psi}\right)\right]^2$$
$$+ \frac{9}{2\kappa(\rho)\rho^2}\left(\frac{\partial S_\psi}{\partial \psi}\right)^2 .$$

Complete integrability on resolved metric cone (8)

Small resolution (7)

As before, ϕ_1, ϕ_2, ψ are cyclic coordinates and the evaluation of $S_{\phi_1}, S_{\phi_2}, S_\psi$ proceeds as before. Fortunately, the evaluation of S_{θ_1} and S_{θ_2} is again as above.

Concerning the radial part of the Hamilton's principal function we get a more intricate equation:

$$E = \frac{1}{2}\kappa(\rho)\left(\frac{\partial S_\rho}{\partial \rho}\right)^2 + \frac{9}{2\kappa(\rho)\rho^2} I_\psi^2 + \frac{3}{\rho^2}\left[(I_{\theta_1} + I_{\phi_1})^2 - I_\psi^2\right]$$
$$+ \frac{3}{(\rho^2 + 6a^2)}\left[(I_{\theta_2} + I_{\phi_2})^2 - I_\psi^2\right] .$$

This equation can be integrated, but the result is not at all illuminating to be produced here.

We remark the asymmetry between the contribution to the Hamiltonian of the action variables associated with the motions on the two sphere S^2. This contrasts with the situation of the geodesic flow on the metric cone of $T^{1,1}$.

Complete integrability on resolved metric cone (9)

Deformation (1)

The deformation of the conifold consists in replacing the apex by an S^3 which is achieved by another modification of equation of the quadric. The metric cone is deformed to a smooth manifold described by the equation:

$$\sum_{a=1}^{4} w_a^2 = \epsilon^2 \,,$$

where ϵ is the “deformation” parameter. Equation $det\, W = 0$ becomes

$$det\, \mathcal{W} = -\frac{1}{2}\epsilon^2 \,.$$

Set the new radial coordinate

$$r^2 = \epsilon^2 \cosh \tau \,,$$

Complete integrability on resolved metric cone (10)

Deformation (2)

Deformed conifold metric is:

$$ds^2_{dc} = \frac{1}{2}\epsilon^{\frac{4}{3}} K(\tau) \left(\frac{1}{3K^3(\tau)} (d\tau^2 + ds_1^2) + \frac{\cosh\tau}{2} ds_2^2 + \frac{1}{2} ds_3^2 \right) ,$$

where

$$K(\tau) = \frac{(\sinh 2\tau - 2\tau)^{\frac{1}{3}}}{2^{\frac{1}{3}} \sinh\tau} ,$$

and

$$\begin{aligned} ds_1^2 &= (d\psi + \cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\phi_2)^2 , \\ ds_2^2 &= d\theta_1^2 + d\theta_2^2 + \sin^2\theta_1 \, d\phi_1^2 + \sin^2\theta_2 \, d\phi_2^2 , \\ ds_3^2 &= 2\big(\sin\psi(d\phi_1 \, d\theta_2 \, \sin\theta_1 + d\phi_2 \, d\theta_1 \, \sin\theta_2) \\ &\qquad + \cos\psi(d\theta_1 \, d\theta_2 - d\phi_1 \, d\phi_2 \, \sin\theta_1 \, \sin\theta_2)\big) . \end{aligned}$$

Complete integrability on resolved metric cone (11)

Deformation (3)

In the limit $r \to \epsilon$, on surfaces $r^2 = \text{const.}$, the deformed conifold metric reduces to the S^3 surface metric.

The coordinate ψ ceases to be a cyclic coordinate and only ϕ_1 and ϕ_2 continue to be cyclic. Therefore the number of the first integrals of the corresponding Hamiltonian is insufficient to ensure the integrability of the geodesic flow.

Outlook

- Sasaki-Ricci flow/soliton
- Deformations of Sasaki structures
- Contact Hamiltonian dynamics on higher dimensional toric Sasaki-Einstein spaces
- Time-dependent Hamilton function
- Dissipative Hamiltonian systems