# *Integrable Structures in Low-dimensional Holography and Cosmology*

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- Conclusions

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The first questions to ask:

• How to match the degrees of freedom on both sides of duality ?

• How exactly the information from the bulk is encoded in the boundary theory ?

## Ryu-Takayanagi holographic entanglement

- Let us have a CFTin a state $|\Psi\rangle$ defined on a spacetime geometry $\mathcal{B}$. Suppose the state $|\Psi\rangle$ is associated with the geometry of a dual theory in a space $M_\Psi$ whose boundary is $\mathcal{B}$.

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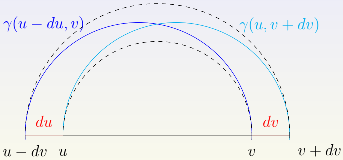
• The surface $\tilde{A}$ is co-dimension 2 extremal surface with the same boundary as $A$!
• The surface $\tilde{A}$ is homologous to $A$, where $\bar{A} \cup A$ is a boundary of d-dimensional space-like region in $M_\Psi$!

When the Ryu-Takayanagi formula applies, in 2d $S(u,v)$ is the entanglement entropy of the interval $(u,v)$.

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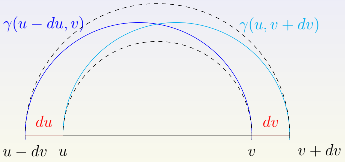


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For intervals

$$A = (u-du, u) \quad \text{and} \quad B = (u,v) \quad \text{and} \quad C = (v, v+dv),$$

strong subadditivity leads to:

$$S(u-du,v) + S(u,v+dv) - S(u,v) - S(u-du,v+dv) \approx \frac{\partial^2 S(u,v)}{\partial u \partial v} \geq 0. \quad (2)$$

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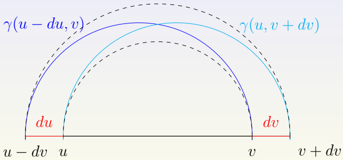


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• Here $S(u,v)$ is the length of the geodesic connecting the boundary points $(u,v)$ (on the cutoff surface).

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Using the replica trick method, the Rényi entropy for the vacuum is given by

$$\exp\left((1-n)S^{(n)}\right)=\langle\Phi_+(z_1)\Phi_-(z_2)\rangle=\frac{1}{(z_1-z_2)^{2h_n}},$$

where twist operators $\Phi_\pm(z)$ have dimensions
$(h_n,\bar{h}_n)=c/24(n-1/n,n-1/n)$.
The entanglement entropy: taking the limit $n\to 1$ of $S^{(n)}$

$$S_{vac}=\lim_{n\to 1}S^{(n)}=\lim_{n\to 1}\log(z_1-z_2)^{-2h_n}=\frac{c}{12}\log\frac{(z_1-z_2)}{\delta^2}.$$

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# A second way: Wilson line anchored at the bdy

The logic of the considerations:
$e^{S_{EE}} = G(\text{geodesic length}) = \text{Wilson line} = \langle \text{mat. element} \rangle$,
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The computations go as follows:

- Use CS formulation of 3d gravity ;
- Choose a convenient basis. In our case we choose

$$L^1 \cong L_{-1} = \partial_x;\ L^0 \cong L_0 = x\partial_x + h;\ L^{-1} \cong L_1 = \frac{1}{2}x^2\partial_x + hx,$$

acting on holomorphic functions of the auxiliary variable $x$ in representation of spin $h$ with $A_{z|y=0} = L^1 + \frac{12}{c}T(z)L^{-1}$.

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- Define a generic Wilson line in this setup

$$W_h(z_f; z_i) = \int dx\,|h\rangle P\left\{e^{\int_{z_i}^{z_f} dz A_z^a(z) L_x^a}\right\}\langle x|.$$

- Wilson lines, bOPE and OPE blocks: (see 1612.06385)

## Matrix elements of an open Wilson line with primary operators at the endpoints

At general c, a Wilson line with primary endpoints can be written in the compact form

$$\langle h|W(z_f,z_i)|h\rangle=\left(e^{\int_{z_i}^{z_f} dz\,\frac{12T(z)}{c}x_T(z)}\frac{1}{x_T(z_i)^2}\right)^h,$$

subject to

$$-x_T'(z)=1+\frac{6T(z)}{c}x_T^2(z),\qquad x_T(z_f)=0,\quad (4)$$

where the function $x_T(z)$ is defined by this differential equation.

The uniformizing $w$-coordinates are connected to the Wilson line

$$\frac{1}{x_T(z)} = \frac{w''(z)}{2w'(z)} - \frac{w'(z)}{w(z)+C},$$

with bdy condition $C = -w(z_f)$.
We therefore find

**Conformal block in the presence of heavy state using Wilson line)**

$$\langle h|W(z_f,z_i)|h\rangle = \lim_{C\to -w(z_f)} \left( e^{-2\int_{z_i}^{z_f} dz \frac{x_T'(z)+1}{x_T(z)}} \frac{1}{x_T(z_i)^2} \right)^h = \left( \frac{w'(z_f)w'(z_i)}{(w(z_f)-w(z_i))^2} \right)^h,$$

exactly reproducing the vacuum Virasoro block for an arbitrary heavy background .

- In the case of Higher spin theories an approach based on skew-tau functions has been used, see 1602.06233.

- The entanglement entropy is given by

$$S_{ex} = \lim_{n\to 1} S_{ex}^{(n)} = \frac{c}{12} \log \left| \frac{f'(z_1) f'(z_2)\, \delta^2}{(f(z_1) - f(z_2))^2} \right| . \tag{5}$$

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- The difference between vacuum entanglement and that of excited states is

$$S_{vac}-S_{ex}=\frac{c}{12}\log\left|\frac{f'(z_1)f'(z_2)\bar{f}'(\bar{z_1})\bar{f}'(\bar{z_2})(z_1-z_2)^2}{(f(z_1)-f(z_2)^2(\bar{f}(\bar{z}_1)-\bar{f}(\bar{z}_2))^2}\right|$$

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- Direct calculations show that $(f'(z)\neq 0)$ the expansion about $z$ is

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$$\frac{f'(z) f'(w)}{(f(z) - f(w))^2} = \frac{1}{(z-w)^2} + \frac{1}{6} S(f)(z) + \frac{1}{12} S'(f)(z)(z-w) + \dots \qquad (6)$$

where $S(f)$ denotes the Schwarzian derivative.
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⇒ Next issue: study of the detailed structure of (5)

• A simple observation: the *exact* expression for entanglement entropy satisfies the Louville field equation:

$$\boxed{\delta^2 \frac{\partial^2 S_{ex}(f)}{\partial u \partial v} = \frac{c}{6} \exp\left(-\frac{12}{c} S_{ex}(f)\right).} \tag{7}$$

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• Let $f$ be a nonconstant meromorphic function on a domain $D$ in the complex plane. For $z \in D$ with $f(z) \neq \infty$, $f'(z) \neq 0$, we consider the quantity

$$G(z+w, z) = \frac{f'(z)}{f(z+w) - f(z)} = \frac{1}{w} - \sum_{n=1}^{\infty} \psi_n[f](z) w^{n-1}.$$

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• The Aharonov invariants are all invariant under global (Möbius) transformations.

# Aharonov invariants and $S_{ex}(f)$

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is invariant under Möbius transformations $\psi_n[M\circ f] = \psi_n[f], \ \ n \geq 2$. The expression entering $S_{ex}$ has the expansion:

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## Recursion relations for Aharonov invariants

The first two $\psi_n[f]$ are

$$\psi_2[f] = \frac{1}{6}\left[\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right] = \frac{1}{3!}S(f). \quad (9)$$

Aharonov proved the recursion formula:

$$(n+1)\psi_n[f]=\psi_{n-1}[f]'+\sum_{k=2}^{n-2}\psi_k[f]\psi_{n-k}[f], \quad n\geq 3. \quad (10)$$

For instance, first few invariants are

$$\psi_3[f] = \frac{S(f)'}{4!}; \quad \psi_4 = \frac{1}{5!}[S''(f) + \frac{2\,S^2(f)}{3}];$$
$$\psi_5 = \frac{1}{6!}[S'''(f) + 3\,S(f)S'(f)]. \quad (11)$$

# Classes univalent functions and Grunsky coefficients

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- The classes univalent functions we use are

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- The functions analytic in $(\infty,\infty)$, $(\infty,0)$ and $(0,0)$:

$$\log\frac{g(z)-g(\zeta)}{z-\zeta},\quad \log\frac{g(z)-f(\zeta)}{z-\zeta},\quad \log\frac{f(z)-f(\zeta)}{z-\zeta}$$

- Another definition ( $\Phi_0(w)\equiv 1$):

$$\frac{g'(z)}{g(z)-w}=\sum_{n=0}^{\infty}\Phi_n(w)z^{-n-1},\quad \Phi_n(w)=\sum_{m=0}^{n} b_{n,m}w^m.$$

$b_{n,m}$ are called Grunsky coefficients.

- Expansions:

$$\log \frac{g(z)-g(\zeta)}{z-\zeta} = -\sum_{m,n=1}^{\infty} b_{mn} z^{-m}\zeta^{-n},$$

$$\log \frac{g(z)-f(\zeta)}{z-\zeta} = -\sum_{m=1,n=0}^{\infty} b_{m,-n} z^{-m}\zeta^{n},$$

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$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{m=0,n=0}^{\infty} b_{-m,-n} z^{m} \zeta^{n}.$$

- Entanglement entropy:

$$\frac{f'(z) f'(w)}{(f(z) - f(w))^2} - \frac{1}{(z-w)^2} = \frac{\partial^2}{\partial z \partial w} \log \frac{f(z) - f(w)}{z - w}$$
$$= -\sum_{m,n\geq 1} mn\, b_{mn} z^{m-1} w^{n-1}. \quad (12)$$

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- Briefs on dToda hierarchy

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and consider

$$W = \lim_{m\to\infty} W_m = 1 + w_1\partial^{-1} + w_2\partial^{-2} + w_3\partial^{-3} + \cdots , \tag{14}$$

where $w_j(j = 1, 2, \dots)$ are functions of $(x, t)$.

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where $w_j (j = 1, 2, \dots)$ are functions of $(x, t)$.

✓ Define the Lax operator

$$L = W\partial W^{-1} = \partial + \sum_{i=1}^{\infty} u_i\partial^{-i+1}, \qquad L^n = W\partial^n W^{-1}, \qquad (15)$$

Define

$$B_n = L^n + B_n^- = \left(W\partial^n W^{-1}\right)^+. \qquad (16)$$

The Lax equation is

$$\frac{\partial L}{\partial t_n} = [B_n, L] = B_nL - LB_n. \qquad (17)$$

## Dispersionless Toda hierarchy

**Definition** The dispersionless Toda hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \mathcal{L}\}, \qquad (18)$$

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where $\mathcal{L}$ and $\bar{\mathcal{L}}$ are generating functions of unknowns $u_i = u_i(t, \bar{t})$, $\bar{u}_i = \bar{u}_i(t, \bar{t})$,

$$\mathcal{L} = p + u_1 + u_2 p^{-1} + u_3 p^{-2} + \cdots \qquad (20)$$

$$\bar{\mathcal{L}} = \bar{u}_0 p^{-1} + \bar{u}_1 + \bar{u}_2 p + \bar{u}_3 p^2 + \cdots \qquad (21)$$

and $\mathcal{B}_n$, $\bar{\mathcal{B}}_n$ are defined by

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}\,, \qquad \bar{\mathcal{B}}_n = \left(\bar{\mathcal{L}}^{-n}\right)_{\leq 0}. \qquad (22)$$

• The equation (22) $\Longrightarrow$ map to n-th Faber polynomial (in certain basis)! The Grunsky coefficients can be identified as $b_{nm} = 1/nm(\partial_n v_m)$ and can be represented in terms of tau-function ( $\mathcal{F} = \log \tau_{dToda}$)!

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**Tau-function and Grunsky coefficients**

The Grunsky coefficients $b_{nm}$ of the pair $(g = w(\mathcal{L}),\ f = w(\bar{\mathcal{L}}))$ are related to the tau function, or fee energy as follows:

$$b_{00}=-\frac{\partial^2\mathcal{F}}{\partial t_0^2},\quad b_{n,0}=\frac{1}{n}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_n},\quad b_{-n,0}=\frac{1}{n}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_{-n}},\quad n\geq 1$$

$$b_{m,n}=-\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_n}\qquad b_{-m,-n}=-\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_{-n}},\quad n,m\geq 1$$

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$$b_{-m,n}=b_{n,-m}=-\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_n}, \quad n,m\geq 1.$$

- The entanglement entropy takes the form

$$S_{vac}-S_{ex}=\frac{c}{12}\log\left(1+(z-w)^2\sum_{m,n}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_n}z^{m-1}w^{n-1}\right).$$

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# The tau-function

✓ The structures appeared so far - $SL(2)$ projective invariants (Schwarzian, Aharonov invariants);

• Next issue: generalization to higher invariants;

- Starting point is again the expression:

$$W_m \partial^m h_0^{(j)}(x;t) = (\partial^m + w_1(x;t)\partial^{m-1} + \cdots + w_m(x;t))\, h_0^{(j)}(x;t) = 0, \quad j = 1, 2, \ldots, m. \tag{23}$$

One can find the expressions for $w_j(x;t)$ as

$$w_j(x;t) = \frac{\begin{vmatrix} h_{m-1}^{(1)} & \cdots & -h_m^{(1)} & \cdots & h_0^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ h_{m-1}^{(m)} & \cdots & -h_m^{(m)} & \cdots & h_0^{(m)} \end{vmatrix}}{\begin{vmatrix} h_{m-1}^{(1)} & \cdots & h_{m-j}^{(1)} & \cdots & h_0^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ h_{m-1}^{(m)} & \cdots & h_{m-j}^{(m)} & \cdots & h_0^{(m)} \end{vmatrix}}. \tag{24}$$

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$$\tau(x;t) = \begin{vmatrix} h_0^{(1)} & \dots & h_0^{(m)} \\ h_1^{(1)} & \dots & h_1^{(m)} \\ \vdots & \dots & \vdots \\ h_{m-1}^{(1)} & \dots & h_{m-1}^{(m)}. \end{vmatrix} \qquad (25)$$

where

$$h_0^{(j)}(x;0) = f^{(j)}(x), \qquad (26)$$

and one can think of $h_n^{(j)}(x;t)$ as defined by

$$h_n^{(j)}(x;t) = \frac{\partial h_0^{(j)}(x;t)}{\partial t_n} = \frac{\partial^n h_0^{(j)}(x;t)}{\partial x^n}. \qquad (27)$$

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• Relations to $w_j$

$$w_j = (-1)^j \frac{1}{\tau} S_{\Box}(\tilde{\partial}_t)\tau. \tag{28}$$

The projective invariants associated to the ordinary differential equation

$$y^{(n)} + p_{n-2}(z)y^{(n-2)} + \cdots + p_0(z)y = 0, \qquad (29)$$

are given by

$$p_i \equiv q_i = \frac{1}{W_n \sqrt[n]{W_n}} \left[ \sum_{j=0}^{n-1} (-1)^{2n-j} (1-\delta_{nj}) \binom{n-j}{n-j-i} \right. \\ \left. . W_{n-j} \left( \sqrt[n]{W_n} \right)^{(n-j-i)} \right], \qquad (30)$$

for $i = 0, 1, \ldots, n-2$.

Here:

$$\tilde{W}_i = \begin{vmatrix} f_1' & f_2' & \cdots & f_{n-1}' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(i-1)} & f_2^{(i-1)} & \cdots & f_n^{(i-1)} \\ f_1^{(i+1)} & f_2^{(i+1)} & \cdots & f_n^{(i+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n)} & f_2^{(n)} & \cdots & f_{n-1}^{(n)} \end{vmatrix}, \qquad W_i = (-1)^{n+i}\tilde{W}. \tag{31}$$

Examle: Let us apply formula (21) to the $n = 2$ case - we have only one invariant, namely $p_0$ which is given by

$$\begin{aligned} p_0 &= \frac{1}{W_2\sqrt{W_2}}\left[W_2\left(\sqrt{W_2}\right)'' - W_1\left(\sqrt{W_2}\right)'\right] \\ &= \frac{1}{2}\left[\frac{f'''}{f'} - \frac{2}{3}\frac{f''^2}{f'^2}\right] \equiv \frac{1}{2}\{f, z\}. \end{aligned} \tag{32}$$

- For completeness, here are the invariants in the case of $n = 3$. This case corresponds to the third order equation

$$y''' + p_1(z)y' + p_0(z)y = 0. \tag{33}$$

The formula (21) gives

$$p_0 = -\frac{1}{3}\left[\frac{2}{9}\left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)^3 - \left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)'' - \left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)\left(\frac{f_1''f_2''' - f_1'''f_2''}{f_1'f_2'' - f_1''f_2'}\right)\right], \tag{34}$$

$$p_1 = \frac{f_1''f_2''' - f_1'''f_2''}{f_1'f_2'' - f_1''f_2'} + \left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)' - \frac{1}{3}\left(\frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}\right)^2,$$

or

$$p_0 = \frac{1}{3}\left[\omega_1\omega_2 - \omega_2'' - \frac{2}{9}\omega_2^3\right], \quad p_1 = \omega_1 + \omega_2' - \frac{1}{3}\omega_2^2.$$

where

$$\omega_1 = \frac{W_1}{W_3} = \frac{f_1'''f_2'' - f_1''f_2'''}{f_1'f_2'' - f_1''f_2'}, \qquad \omega_2 = \frac{W_2}{W_3} = \frac{f_1'f_2''' - f_1'''f_2'}{f_1'f_2'' - f_1''f_2'}$$

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- From $SL(2)$ formulation to $SL(n)$ - define the connections as

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• Go to the main goal - bulk reconstruction!

# The setup: radial evolution

- Consider Lorentzian $d+1$ dimensionsl manifold $(\mathcal{M}, g)$ which is solution of the Einstein equation.

*Definition*: The manifold $(\mathcal{M}, g)$ is called conformally compact if $\exists$ a defining function

$$\rho^{-1}(0) = \partial\mathcal{M}, \qquad \partial\rho \neq 0 \text{ on } \partial\mathcal{M}, \tag{35}$$

& the conf. equiv. metric $\ell^2 \bar{g} = \rho^2 g$ extends smoothly on $\partial\mathcal{M}$.

- Let $\partial\mathcal{M} = \Sigma$. At some $\rho$ we have $\Sigma_\rho$ fro which we have

$$n = \partial_r = -\frac{\rho}{\ell}\partial_\rho, \quad K^\nu_\mu = \gamma^{\nu\alpha}\nabla_\alpha n_\mu = \frac{1}{2}\gamma^{\nu\alpha}\mathcal{L}_v g_{\alpha\mu}$$
$$\gamma_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \quad \varepsilon = n^2, \quad \vec{n} \perp \Sigma_\rho. \tag{36}$$

- Radial evolution

$$\partial_r \Psi(\gamma_\rho) = \int_{\Sigma_\rho} \partial_r \gamma_{ij} \frac{\delta\Psi}{\delta\gamma_{ij}} \underset{\rho=0}{\sim} \frac{2}{\ell}\int_{\Sigma_\rho} \gamma_{ij}\frac{\delta\Psi}{\delta\gamma_{ij}}, \tag{37}$$

where the last operator is just the operator of conformal scaling. This means that the radial evolution is intimately related to the conformal rescaling.

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- Gauge invariance (2d)

$$H_v \Psi(A) \equiv \int_\Sigma \left( v A^a + \frac{\pi}{k} \frac{\delta}{\delta A}^a \right) \left( \bar{\partial} A^a - \frac{\pi}{k} \partial \frac{\delta}{\delta A}^a \right) \Psi(A) = 0.$$

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- Make Fourier-Laplace transformation:

$$\Psi(\mu)=\int DA\Psi(A)\chi_\mu(A),\qquad \chi_\mu=e^{-\frac{k}{2\pi}\int\mu\,\mathrm{tr}\,A^2}. \tag{38}$$

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- The $H_v$ constraint in $\mu$-representation $\Longrightarrow$ Ward Identity

$$H_v\Psi(A) = 0 \ \Rightarrow \int_{\Sigma_\rho} (v + \mu\bar v)\left(\bar\partial - \mu\partial - 2\partial\mu\right)\frac{\delta}{\delta\mu(z)}\Psi(A)$$

$$= -\frac{c}{12\pi}\int_{\Sigma_\rho} d^2z(v + \mu\bar v)\partial^3\mu\Psi(A),$$

## Wheeler-deWitt equation and Ward Identity

- Define for 3d *gravity* case the conjugate

$$\Pi^{ab} = \frac{2}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{ab}}, \qquad \Pi = \pi_a^a,$$

- Diffeo's

$$H_b = \nabla_a \Pi_b^a.$$

- The Hamiltonian constraint (geometry idependence, AdS case)

$$H = \kappa^2 : \left(\Pi^{ab}\Pi^{cd}G_{abcd} - \frac{\Pi^2}{d-1}\right) : + R(\gamma) + \frac{d(d-1)}{l^2},$$

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implies "Wheeler-deWitt equation!

$$H\Psi = 0.$$

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- For general $n$ ($W_n$ case)

$$\mathcal{L} \sim \sum_i \text{coeff}_i \, \text{tr} \, q_i\bar{q}_j + \cdots .$$

# A sketch of relation to SYK

- The Sachdev-Ye-Kitaev (SYK) model describes interacting Majorana fermions with random (gaussian) coupling

$$S = \frac{i}{2}\int dt \left( \sum_{\alpha=1}^{N} \psi^{\alpha}\partial\psi_{\alpha} - i^{q} \sum_{\alpha_1\ldots\alpha_q} J^{\alpha_1\ldots\alpha_q}\psi_{\alpha_1}\ldots\psi_{\alpha_q}, \right)$$

where

$$\langle J^{\alpha_1\ldots\alpha_q} J^{\beta_1\ldots\beta_q}\rangle = \frac{J^2(q-1)}{N^{q-1}}\prod_{i}^{q}\delta^{\alpha_i\beta_i}.$$

- SYK model addresses many interesting issues as properties of non-Fermi liquid behavior, quantum chaos, emergent conformal symmetry and holographic duality. SYK model can be used to describe black holes (BHs) in 2d nearly-Anti-de-Sitter gravity.
- The effective action is just the Schwarzian

$$S_{Sch} = -C\int td\{f,t\}, \qquad \{f,t\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

• 2d generalization suggested in 1701.00528 (G. Turiaci, H, Verlinde), leading to double Schwarzian theory in the UV (in light-cone)

$$S_{UV} \sim \int du\, dv \{x_+, u\}\{x_-, v\}.$$

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• Using Lagrange multipliers $\Longleftrightarrow$

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• Relations between double Schwarzian & Polyakov-Liouville actions ($\mathcal{L} \sim p_0 \bar{p}_0$ in our case) - in 1701.00528 (G. Turiaci, H, Verlinde)

$$S_{grav}[E] = \min_e \left( \int S_L(E+e) - \int \epsilon^{\mu\nu} e_\mu^+ e_\nu^- \right).$$

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• *We generalize the above picture to arbitrary higher spin theories by making use of W-geometry and jet bundles*!

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  - . . . and much more . . .

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*T H A N K Y O U!*