Belgrad School
April 2009

INFLATION

① $\hbar = 0$ Problems with hot Big Bang Cosmology
What is inflation? Why it solves the problems
Slow-roll inflation

② $\hbar \neq 0$ Quantum mechanics with a time-dependent background
Scalar + tensor perturbations : predictions

③ Non-gaussianity Free field theory ⟶ Interactions
Hot topic : present status + future
(Slow-roll eternal inflation)

✓ Very far from predictions of slow-roll inflation

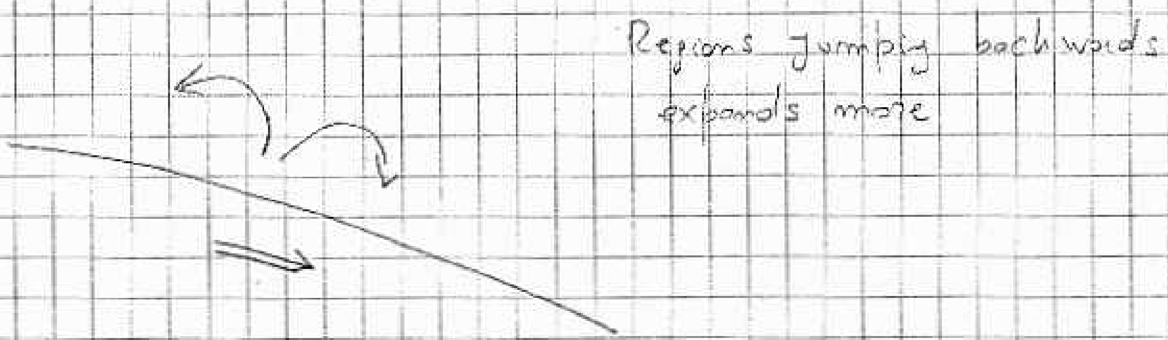
> NG is seen as a smoking gun of something non-minimal going on!

Eternal inflation (slow-roll case)

What happens when $H^2/\dot{\phi} \sim 1$?

Notice $H^2/\dot{\phi}$ depends on the position along the potential and in many cases it becomes $O(1)$ somewhere ...

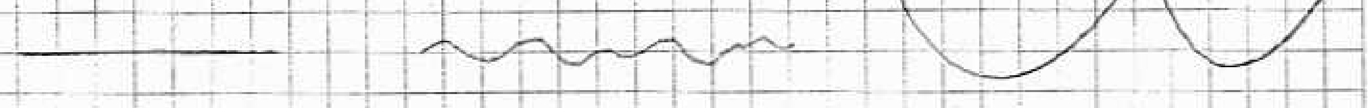
→ You still have perturbative control (slow-roll expansion)



→ Starting from a Hubble box, in a time H^{-1} you generate $e^3 \sim 20$ Hubble boxes. If the probability of jumping backward $\gtrsim \frac{1}{20}$ I expect the process to never terminate

$\rightsquigarrow$ Eternal inflation for $H^2/\dot{\phi} \sim 1$

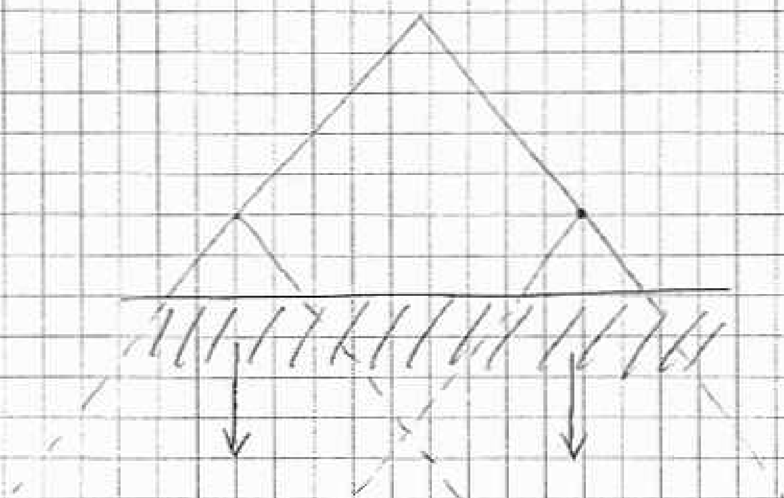
→ More precisely one can study the probability distribution of the reheating volume



Various versions of the same problem: curvature, monopole ...

(Guth '81)

• How do we solve this problem?



In picture it is simple ...

We have to make the η integral large

$$d_{HOR} = a \int_0^t \frac{d\tilde{t}}{\tilde{t}^{q}}$$

For $q > 1$ the integral diverges in the past

$d_{HOR} \neq H^{-1}$!

t^q $\quad q > 1 \quad$ has $\quad \ddot{a} > 0$

This is general: problem is solved for $\ddot{a} > 0$

$$\boxed{\text{INFLATION} \equiv \ddot{a} > 0}$$

With this definition we are in a period of inflation ...

Dimensionless $V'''' \lesssim O(\epsilon^2; \eta^3) \dfrac{H^6}{V'^2} (10^{-5})^2$

The inflaton is <u>incredibly</u> weakly coupled. Cfr Higgs which has $\lambda \sim 1$

- I <u>have</u> to take into account the mixing with gravity

$$S = \int \sqrt{-g} \left\{ \frac{M_P^2}{2} R - \frac{1}{2} (\nabla\phi)^2 - V(\phi) \right\}$$

$$\begin{cases} ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt) \\ h_{ij} = a^2(t) \left[e^{2\zeta} \delta_{ij} + h_{ij} \right] \\ \delta\phi = 0 \end{cases}$$

graviton modes

N and N^i are not dynamical, no time derivatives: solve for them and plug them in

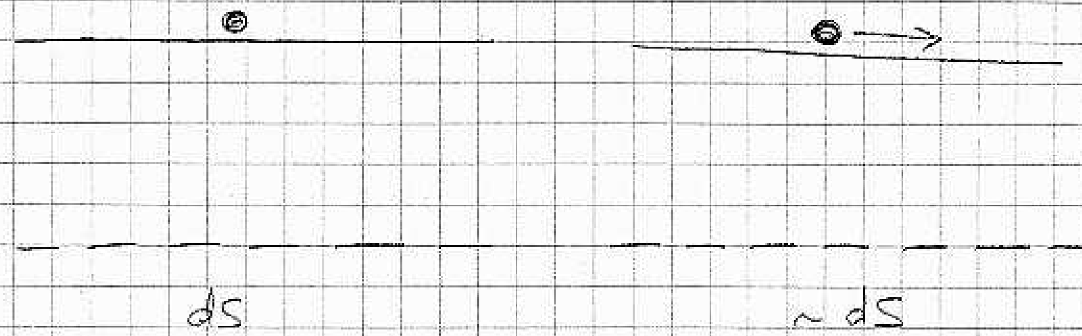
$$S_2 + S_3 = \int d^4x \, a^3 \frac{\dot\phi^2}{H^2} \left[\dot\zeta^2 - \left(\frac{\partial}{a} \zeta \right)^2 \right] + a^3 \frac{\dot\phi^2}{2H^2} \left[2 \frac{\partial_i}{\partial^2} \dot\zeta \, \partial_i \zeta \, \frac{\partial^2}{a^2} \frac{\zeta}{H} \right] + \text{many others}$$

Evaluating all derivatives at horizon-crossing $\partial_i \sim \partial_t \sim H$, we have a suppression $O(\zeta) \simeq 10^{-5}$

Actually there is an additional $O(\epsilon; \eta)$ suppression

- Most successful candidate : slow-roll inflation

Very profound



Take the lagrangian of a minimally coupled scalar (can always do a conformal transformation)

Inflaton

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi)$$
$$p = \frac{1}{2} \dot{\phi}^2 - V(\phi)$$

We want to stay close to dS : Kinetic energy small compared to potential

Slow-roll : $\frac{\dot{\phi}^2}{2} \ll V$ (for many Hubble times)

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \\ H^2 = \frac{1}{3M_p^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \end{cases} \longrightarrow \begin{cases} \dot{\phi} \simeq -\frac{V'}{3H} \\ H^2 \simeq \frac{V}{3M_p^2} \end{cases}$$

✓ Perturbative decay of the inflaton: Γ

$\sim$ decays when $H \simeq \Gamma \implies T_{re} \simeq \sqrt{M_P \Gamma}\, g_*^{-1/2}$

✓ Non-perturbative decay: $m^2(\phi)\,\chi\chi$

Reheating is complicated + model dependent + most likely unobservable

- Classification of inflationary models (personal)

1) Nothing abrupt at end of inflation

$$V \propto \phi^n \qquad \epsilon, \eta \simeq \left(\frac{M_P}{\phi}\right)^2$$

$$\epsilon, \eta \sim \frac{1}{N} \qquad \Delta\phi \gg M_P \qquad \boxed{\text{large } H,\ \text{large } \Delta\phi}$$

Given $\frac{1}{\sqrt{\epsilon}} \frac{H}{M_P} \simeq 10^{-5}$ fixed by experiments

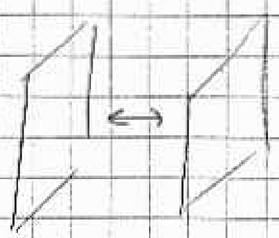
These models have very large H ($\sim 10^{14}$ GeV)

$\frac{H}{M_P}$ gives GW

2) Hybrid models

$$V = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda (\chi^2 - M^2)^2 + \frac{1}{2} \lambda' \chi^2 \phi^2$$

ϕ rolls and triggers a phase transition for χ



E.g. Brane inflation

$H \simeq$ const $\implies \epsilon \lll 1$

- Scale dependence

$$\langle \zeta_{\vec{k}}(\eta)\, \zeta_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') \left. \frac{H^4}{2k^3 \dot{\phi}^2} \right)_{\text{horizon crossing}}$$

H and $\dot{\phi}$ are not exactly constant and this introduces a small scale dependence

$$k^{-3+(n_s-1)}$$

$$n_s - 1 = \frac{d}{d\log k} \log \left. \frac{H^4}{\dot{\phi}^2} \right|_{\text{crossing}} = H^{-1} \frac{d}{dt} \log \frac{H^6}{V'^2}$$

$k \sim aH \quad a \sim e^{Ht}$

$d\log k = H dt$

$$= 6 \frac{\dot{H}}{H^2} - 2 \frac{V'' \dot{\phi}}{V' H} = -6\epsilon + 2\eta$$

Red spectrum : $n_s - 1 < 0$ More power at low k

Blue spectrum : $n_s - 1 > 0$ " at high k

$$n_s = 0.96 \pm 0.02$$

Strong evidence that $n_s \neq 1$!

- Gravity waves. Every light field gets excited

$$\langle \gamma^s_{\vec{k}}\, \gamma^{s'}_{\vec{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') \frac{1}{2k^3} \frac{2H^2}{M_P^2} \delta_{ss'}$$

Suppressed by $\sqrt{\epsilon}$ compared to scalar modes

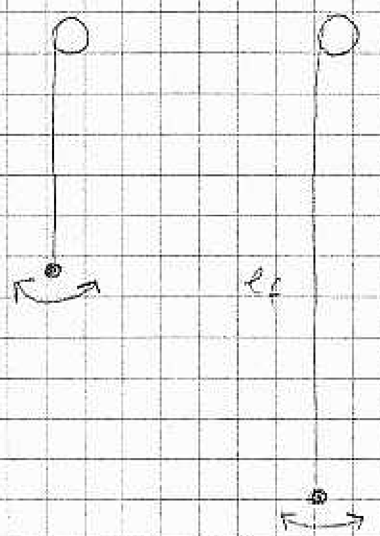
r is the ratio between the two contributions

Lecture 2 : $\hbar \neq 0$

For $\hbar = 0$ we are driven (if inflation lasts enough) to a completely homogeneous + flat Universe → $V \sim$ constant, other things dilute

If we include QM the story is different ... → Modes are stretched out of the horizon

- Harmonic oscillator:



$l_i \longrightarrow l_f$

$\omega_i = \sqrt{\frac{g}{l_i}} \longrightarrow \omega_f = \sqrt{\frac{g}{l_f}}$

✓ If we start at rest, classically nothing happens

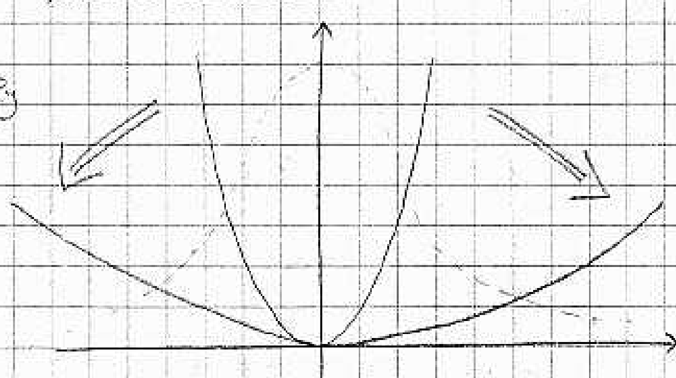
✓ QM, two possible behaviours

→ $T \gg \omega_i^{-1} ; \omega_f^{-1}$ adiabatic limit

We follow the instantaneous vacuum

→ $T \ll \omega_i^{-1} ; \omega_f^{-1}$ sudden limit

The potential varies instantaneously + the wavefunction remains the one of the old vacuum



Excited state of the new h.o.

For $\omega_f \ll \omega_i$ $\quad E_f = \frac{1}{4} \hbar \omega_i$ (I lost the potential energy)

Very excited $\quad N_f = \frac{1}{4} \frac{\omega_i}{\omega_f} \gg 1$

$$\phi(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[\phi_k^{cl}(\eta)\, \hat{a}_k^{+} + \phi_k^{cl\,*}(\eta)\, \hat{a}_k \right] e^{i\vec{k}\vec{x}}$$

What is the 2pf of ϕ in the large scale limit?

$$\langle \phi_{\vec{k}}(\eta)\, \phi_{\vec{k}'}(\eta') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{H^2}{2k^3}$$

People call this kind of spectrum "scale invariant". Why?

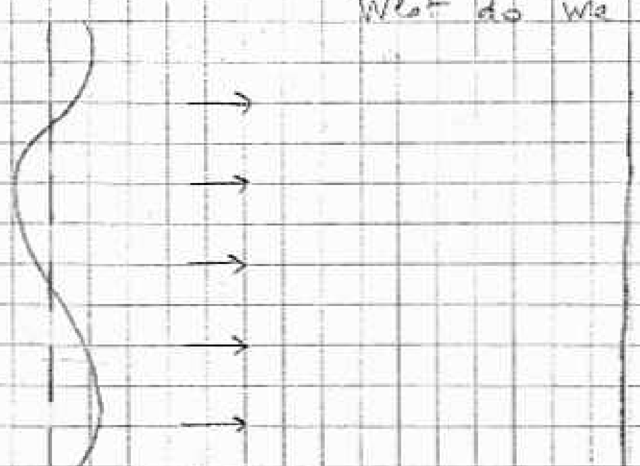
In real space:

$$\langle \phi(x;\eta)\, \phi(y;\eta) \rangle = -\frac{H^2}{(2\pi)^2} \log \frac{|x - x'|}{L}$$

Not only absence of a scale, but a precise dependence on the distance

What do we really measure?

How do we connect with perturbations in the MBB?



end of inflation

Fluctuations $\delta\phi$ change the time to the end of inflation

Extra expansion: $\zeta = H\,\delta t = -\frac{H\,\delta\phi}{\dot{\phi}_0}$ $\quad \boxed{\frac{H^2}{\dot{\phi}} \simeq \frac{H^2}{V'} 3H \simeq \frac{H}{M_P} \cdot \frac{1}{\sqrt{\epsilon}}}$

$$\langle \zeta_{\vec{k}}(\eta)\, \zeta_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \underbrace{\frac{H^2}{2k^3}}_{\text{Free scalar ds}} \underbrace{\left(\frac{H}{\dot{\phi}}\right)^2}_{\text{to } \zeta}$$

ζ describes curvature of constant inflaton surfaces, or equal

Consider each Fourier mode of the inflaton (independent for the time being)

$\omega = \frac{k}{a}$ $\qquad$ H is the scale of the time dependent background

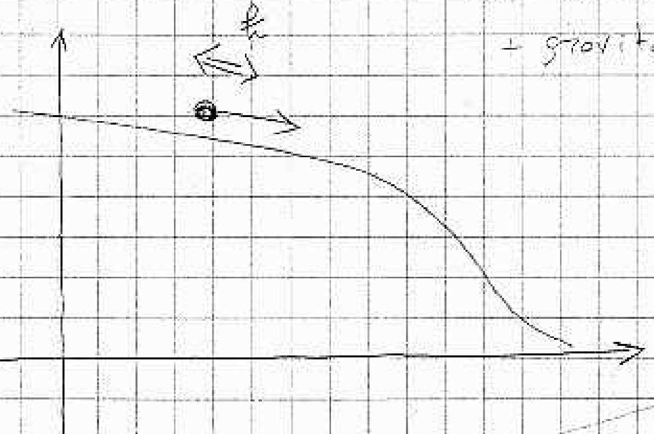
$\frac{k}{a} \gg H$ $\qquad$ inside Hubble radius : adiabatic

$\frac{k}{a} \ll H$ $\qquad$ outside " " : production of fluctuations

Huge squeezing : $\sim a$ between Hubble exit / reentering
$\gtrsim e^{60}$

Fluctuations of the inflaton set initial conditions for subsequent evolution : CMB, LSS .. everything reconstructs the initial perturbations + gravitational instability

$\left(\delta\phi \rightarrow \phi \right)$



$S = \int d^4x \, \frac{1}{2H^2\eta^2} \left[(\partial_\eta \phi)^2 - (\partial_i \phi)^2 \right]$

$\sim dS$ $\quad$ $\sim$ massless $\quad$ Massless scalar in dS

$\eta \in (-\infty, 0)$ $\qquad$ $\phi_k^{cl}(\eta) = \frac{H}{\sqrt{2k^3}} \left(1 - ik\eta \right) e^{ik\eta}$

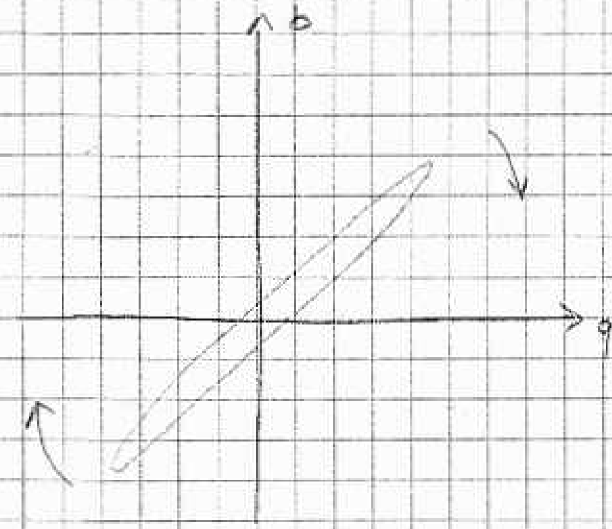
$\begin{cases} k\eta \gg 1 & \text{inside Hubble} \sim \text{oscillates as Minkowski} \\ k\eta \ll 1 & \text{outside Hubble} \longrightarrow \text{constant} \end{cases}$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\,\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right)$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - i\,\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right) \qquad \hat{a}_{OLD}|\psi\rangle = 0$$

Changing ω, changes the linear combination which gives $\hat{a}$ and $\hat{a}^{+}$: Bogoliubov transformation

$$|\text{squeezed}\rangle = e^{\frac{1}{2}\xi^{*}\hat{a}^{2} - \frac{1}{2}\xi\,\hat{a}^{+2}}|0\rangle$$



The uncertainty is "squeezed" in one direction and it becomes very big in the other

For large squeezing we can forget about the squeezed direction : the e.o. has a particular phase and a large macroscopic uncertainty in the amplitude : classical, stochastic variable

- $$\phi(t;\vec{x}) = \phi_0(t) + \delta\phi(\vec{x};t)$$

Classically these perturbations are just diluted and driven to zero by the expansion.

But QM ...

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\underset{\ll H^2}{V''} + \frac{k^2}{a^2}\right)\delta\phi_k = 0$$

- ζ is conserved out of the Hubble radius independently of what happens when a mode is out (phase transitions, reheating, how many dof? ...)
  Good and bad ...

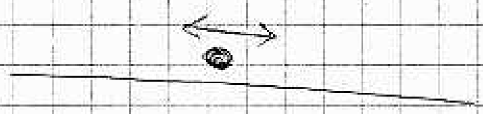
Intuitively, parallel Universes:

$$ds^2 = -dt^2 + e^{2\zeta(x)} a^2(t)\, d\vec{x}^{\,2}$$

The evolution is the same everywhere, with a relative extra expansion, which thus stays constant

- Inflaton fluctuations change also the metric, why I did not take this into account?

Subleading for $\epsilon, \eta \to 0$ (in fact gauge)



For $\epsilon \to 0$ the metric does not change much.

This does not suppress, but enhances $\frac{H^4}{\dot\phi^2}$ as $\dot\phi \to 0$

- $\langle \zeta \zeta \rangle \simeq \frac{H^4}{\dot\phi^2} \simeq (10^{-5})^2$

$$\boxed{\frac{\delta\rho}{\rho} \simeq \frac{V'\delta\phi}{V} \simeq \frac{H}{M_P}\sqrt{\epsilon}}$$

experimentally, $\frac{\delta T}{T}$ eg

$$\frac{H^2}{\dot\phi} \sim \frac{H}{M_P}\frac{1}{\sqrt{\epsilon}}$$

ratio between quantum and classical fluctuations in 1 Hubble time

$$\Delta\phi \simeq \frac{\dot{\phi}}{H} \simeq \frac{V'}{H^2} M_P^2 \simeq \sqrt{\epsilon}\, M_P \ll M_P$$

$$\frac{1}{\sqrt{\epsilon}} \frac{H}{M_P} \text{ is fixed} \longrightarrow H \text{ is small} \Longrightarrow \frac{H}{M_P} \ll \frac{1}{\sqrt{\epsilon}} \frac{H}{M_P}$$

No GWs

Which one is "generic"?

$\boxed{\text{Small } H, \text{ small } \Delta\phi}$

Measurable only if $r \gtrsim 0.01$

Observation of GW : $\frac{H}{M_P}$ is measured + H is large

$$\Delta\phi \simeq \dot{\phi}/_H \cdot N \simeq V'/_{H^2} \cdot N \simeq \sqrt{\epsilon} \cdot M_P \cdot N \qquad \left(\text{Lyth's bound}\right)$$

Gravity waves are a real probe of inflation : they are only sensitive to H + we would probe a field displacement $\gtrsim M_P$!

$$n_t = \frac{d\log H^2}{d\log k} = -2\epsilon \qquad \text{always red as } \dot{H} < 0$$

This enters also in r . Consistency relation

Very hard to probe !

- Slow-roll parameters

$$\frac{1}{2}\dot{\phi}^2 \ll V \qquad \frac{V'^2}{H^2} \ll V$$

$$\epsilon = \frac{1}{2} M_p^2 \left(\frac{V'}{V}\right)^2 \ll 1$$

$$\frac{\dot{H}}{H^2} = -\epsilon$$ H is constant for dS. ϵ gives the departure from dS

$$\ddot{\phi} \simeq \left(\frac{V'}{-3H}\right)^{\cdot} \underset{\text{up to } \epsilon}{\simeq} \frac{V''}{H}\dot{\phi} \simeq \frac{V''V'}{H^2} \ll V'$$

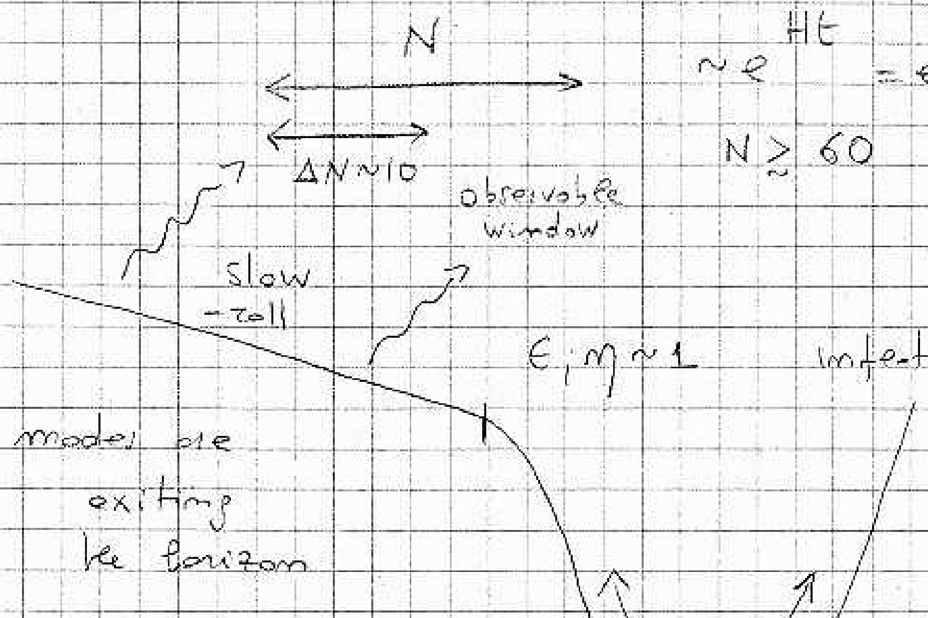
$$\Rightarrow V'' \ll H^2$$ (the mass^2 of inflaton $\ll H^2$)

$$\eta = M_p^2 \frac{V''}{V} \ll 1$$

- Inflationary cosmology

$\sim e^{Ht} = e^N$

$N \gtrsim 60$



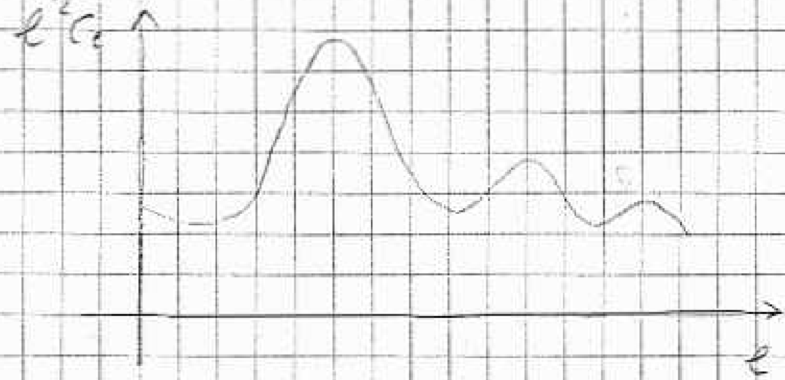
oscillation + transfer of energy to SM

REHEATING

Lecture 3 . Primordial non-Gaussianities

Use of cosmological data to reconstruct initial conditions set by quantum fluctuations during inflation

* Each mode behaves independently, everything is given by the 2pf



Initial conditions + plasma

Information summarized in the spectrum

No statistical correlation among modes $\iff$ Free quantum theory

* Last 6 years a lot of development in theory + experiment

Maldacena 02 $\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle$

The result is VERY small as the inflaton is a very weakly coupled field

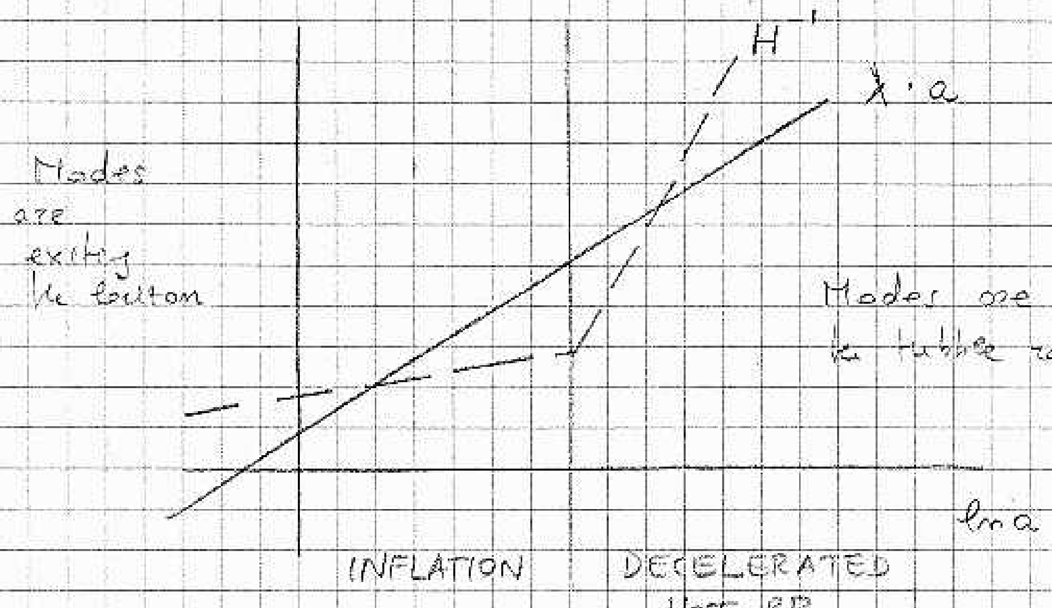
$$\frac{\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle}{\langle \zeta_{k_1} \zeta_{k_1} \rangle^{3/2}} \sim \text{skewness}$$

Inflation $\downarrow$ Flatness $\downarrow$ Weak coupling

$$\frac{V'''}{H} \qquad \eta \simeq \frac{V''}{H^2} \qquad \Delta\eta_{H^{-1}} \supset \frac{V''' V'}{H^4} \ll 1$$

$$\frac{V'''}{H} \simeq O(\epsilon^2; \eta^2) \frac{H^3}{V'} \quad 10^{-5}$$

Let us look at this from another point of view ...



$a \sim t^q \quad \lesssim \quad t \sim H^{-1}$

- How to obtain $\ddot{a} > 0$?

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G (\rho + 3p) > 0 \qquad \text{Gravity is repulsive}$$

A C.C. > 0 is a good candidate

$$T_{\mu\nu} = \begin{pmatrix} \rho = \Lambda & & & \\ & p = -\Lambda & & \\ & & p = -\Lambda & \\ & & & p = -\Lambda \end{pmatrix}$$

In the presence of a $\Lambda > 0$ $\quad a(t) = e^{Ht} \qquad \ddot{a} > 0$

We want this phase to terminate and start the hot BB Cosmology! Graceful exit ...

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) \underbrace{P_\zeta}_{\frac{H^4}{\dot{\phi}^2}} \frac{1}{k_1^3}$$

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \, P_\zeta^2 \, f_{NL} \, F(k_1; k_2; k_3)$$

Homogeneous function of degree -6

With this definition $f_{NL} \sim 1$ means

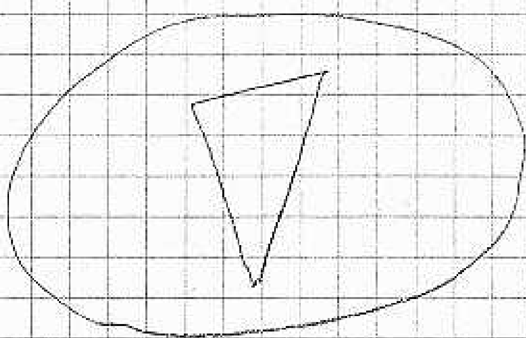
$$\frac{\langle \zeta\zeta\zeta \rangle}{\langle \zeta\zeta \rangle^{3/2}} \sim O(10^{-5})$$

Maldacena's result gives $f_{NL} \sim O(\epsilon)$

* Experimentally

✓ CMB gives the best constraints so far (LSS in the future?)

✓ In first approx CMB reproduces initial conditions



Quantum correlation function in the sky!

✓ $$\frac{\langle \zeta\zeta\zeta \rangle}{\langle \zeta\zeta \rangle^{3/2}} \sim f_{NL} \, 10^{-5} \sim \frac{1}{\sqrt{N_{pixels}}} \sim 10^{-3}$$

$$|f_{NL}| \lesssim 100$$

One of the most known things in cosmology!

Lecture 1

- Homogeneity problem

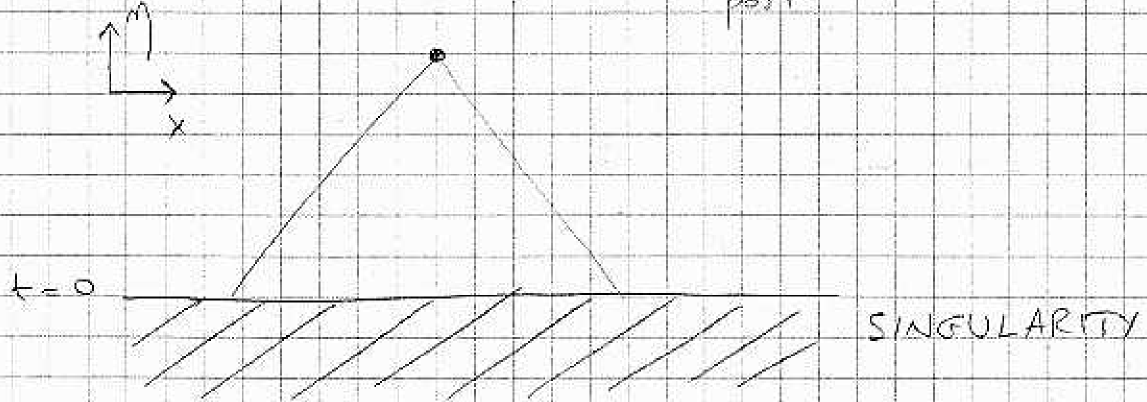
$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \qquad \text{FRW (flat)}$$

$$\text{RD}: \quad a(t) \propto t^{1/2}$$
$$\text{MD}: \quad a(t) \propto t^{2/3} \qquad H = \frac{\dot a}{a} \sim \frac{1}{t}$$

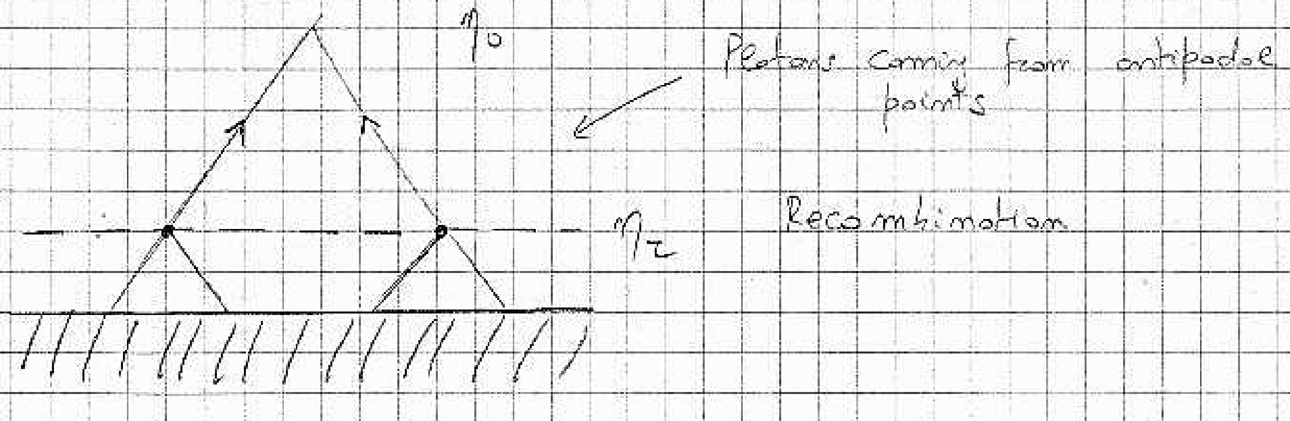
$$ds^2 = a^2(\eta)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right)$$

$$\eta = \int^t \frac{dt}{a(t)}$$

Both in MD and RD the integral converges in the past



$$d_{HOR} = a \int_0^t \frac{d\tilde t}{a(\tilde t)} \sim t \sim H^{-1} \qquad \text{Hubble radius}$$



Why CMB photons have ~ same T, though they have never been in causal contact?

One can show that there is a phase transition in the probability distribution of the volume

$$P\left(V_{\text{reheating}} ; \frac{H^2}{\dot{\phi}} \right)$$

develops a finite probability of ∞ reheating volume

at $\frac{H^2}{\dot{\phi}} = \sqrt{\frac{2}{3}}\, \pi$

landscape, measure problem...