I. BAKAS

Gravitational perturbations, duality and holography

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LECTURE I:

AdS_4 space-time

Electric/Magnetic Duality

Maxwell eqs without sources are

$$\nabla_\mu F^{\mu\nu} = 0 \qquad (1)$$

$$\nabla_\kappa F_{\mu\nu} + \nabla_\mu F_{\nu\kappa} + \nabla_\nu F_{\kappa\mu} = 0 \qquad (2)$$

on a fixed 4-d spacetime M_4.

Using dual field strength

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu}{}_{\kappa\lambda} F^{\kappa\lambda}$$

Second set of equations becomes

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0 \qquad (2)'$$

Interchanging F and $\tilde{F}$ amouts to familiar E/M duality

$$\vec{E} \to \vec{B} \,, \quad \vec{B} \to -\vec{E} .$$

Thus, on a fixed (in generally curved) 4-d spacetime, for a given gauge field A with

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

there is a dual gauge field $\check{A}$ defined as follows,

$$\tilde{F}_{\mu\nu}(A) = F_{\mu\nu}(\check{A}),$$

$$\tilde{F}_{\mu\nu}(\check{A}) = -F_{\mu\nu}(A).$$

The relation among A and $\check{A}$

- is not a gauge transformatin
- it is in general non-local

Electric/Magnetic duality had profound applications.

Gravitational analogue

Consider 4-d space-time M_4 with Lorentzian signature and metric g. The Riemann curvature satisfies:

- translational Biachi identity

$$R_{\mu[\nu\kappa\lambda]} = 0,$$

- rotational Bianchi identity

$$\nabla_{[\rho} R_{\mu\nu]\kappa\lambda} = 0.$$

Einstein eqs without sources are

$$R^{\rho}{}_{\mu\rho\nu} \equiv R_{\mu\nu} = 0$$

assuming for the moment that cosmological constant $\Lambda = 0$

Next, define dual curvature

$$\tilde{R}_{\mu\nu\kappa\lambda} = \frac{1}{2} \varepsilon_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma\kappa\lambda}$$

which also satisfies similar identities (on-shell)

$$\tilde{R}_{\mu[\nu\kappa\lambda]} = 0,$$

$$\nabla_{[\rho} \tilde{R}_{\mu\nu]\kappa\lambda} = 0$$

and

$$\tilde{R}^{\rho}{}_{\mu\rho\nu} \equiv \tilde{R}_{\mu\nu} = 0.$$

Thus, it is natural to ask whether interchanging R and $\tilde{R}$ provides gravitational analogue of electric/magnetic duality.

To mimic Maxwell theory, we choose a fixed reference metric $g^{(0)}$ and expand around it

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

It turns out that $g^{(0)}_{\mu\nu}$ remains inert provided that

$$g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$$

i.e., for small fluctuations around flat space-time.

Then, we have a graviton h

$$R_{\mu\nu\kappa\lambda}(h) = \partial_{[\mu} h_{\nu][\kappa,\lambda]}$$

and a dual graviton $\check{h}$ defined as

$$\tilde{R}_{\mu\nu\kappa\lambda}(h) = R_{\mu\nu\kappa\lambda}(\check{h}),$$

$$\tilde{R}_{\mu\nu\kappa\lambda}(\check{h}) = -R_{\mu\nu\kappa\lambda}(h).$$

Generalization in the presence of cosmological constant ($\Lambda>0$ or <0) is achieved by replacing Riemann curvature with Weyl curvature tensor, which (on-shell) is

$$W_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \frac{\Lambda}{3}\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right).$$

Then,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

where $g^{(0)}_{\mu\nu}$ is metric of dS_4 or AdS_4 space-time depending on sign of Λ.

Thus, "electric/magnetic" duality in linearized 4-d gravity extends to all values of Λ.

For later use, introduce "electric" and "magnetic" compts of Weyl tensor

$$\mathcal{E}_{ab} = W_{arbr} ,$$

$$B_{ab} = \widetilde{W}_{arbr}$$

which are symmetric and traceless. They are appropriate for radial ADM decomposition of 4-d metric.
Then, gravitational duality reads

$$\mathcal{E}_{ab} \rightarrow B_{ab} , \; B_{ab} \rightarrow -\mathcal{E}_{ab} .$$

- is not a coordinate transformation
- it is in general non-local

Need to explore and apply it.

Axial/polar duality

Go back to Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0$$

on fixed spherically symmetric static background

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Expand A_μ in vector spherical harmonics as

Axial

$$A_\mu(t,r,\theta,\phi) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_\ell(r)\sin\theta\, \partial_\theta P_\ell \end{pmatrix} e^{-i\omega t}$$

with parity $(-1)^{\ell+1}$.

Polar

$$A_\mu(t,r,\theta,\phi) = \begin{pmatrix} f_\ell(r) P_\ell \\ h_\ell(r) P_\ell \\ k_\ell(r)\partial_\theta P_\ell \\ 0 \end{pmatrix} e^{-i\omega t}$$

with parity $(-1)^\ell$.

AXIAL

The field strength takes the form

$$F_{t\phi} = -i\omega a_\ell(r) e^{-i\omega t} \sin\theta\, \partial_\theta P_\ell \,,$$

$$F_{r\phi} = a_\ell'(r) e^{-i\omega t} \sin\theta\, \partial_\theta P_\ell \,,$$

$$F_{\theta\phi} = -\ell(\ell+1) a_\ell(r) e^{-i\omega t} \sin\theta\, P_\ell$$

and Maxwell eqs take form of an effective Schrödinger problem

$$\left(-\frac{d^2}{dr_*^2} + V(r)\right)\Psi_{axial}^{(r)} = \omega^2 \Psi_{axial}^{(r)}$$

with respect to tortoise coordinate r_*,

$$dr_* = \frac{dr}{f(r)} \,,$$

where

$$V(r) = f(r)\, \frac{\ell(\ell+1)}{r^2}$$

and

$$\Psi_{axial}^{(r)} = a_\ell(r) \,.$$

POLAR

Here, the field strength is

$$F_{tr} = -\left(f_\ell'(r) + i\omega h_\ell(r)\right) e^{-i\omega t} P_\ell ,$$

$$F_{r\theta} = \left(k_\ell'(r) - h_\ell(r)\right) e^{-i\omega t} \partial_\theta P_\ell ,$$

$$F_{t\theta} = -\left(f_\ell(r) + i\omega k_\ell(r)\right) e^{-i\omega t} \partial_\theta P_\ell$$

and Maxwell eqs take form of an effective Schrödinger problem

$$\left(-\frac{d^2}{dr_*^2} + V(r)\right) \Psi_{polar}(r) = \omega^2 \Psi_{polar}(r)$$

wrt tortoise radial coordinate, where

$$V(r) = f(r) \frac{\ell(\ell+1)}{r^2}$$

is same as before and

$$\Psi_{polar}(r) = r^2 \left(f_\ell'(r) + i\omega h_\ell(r)\right).$$

Rewriting $F_{\mu\nu}$ in terms of $\Psi(r)$ and $\Psi'(r)$, it turns out that

$$F_{\mu\nu}^{polar} = \tilde{F}_{\mu\nu}^{axial} ,$$

$$F_{\mu\nu}^{axial} = - \tilde{F}_{\mu\nu}^{polar}$$

provided that

$$\Psi_{polar}(r) = \ell(\ell+1)\Psi_{axial}(r) .$$

Thus, electric/magnetic duality of electromagnetism manifests as axial/polar duality provided that the two effective Schrödinger problems satisfy the same boundary conditions (so that Ψ and ω are the same).

Gravitational perturbations

Consider gravitational perturbation

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \equiv \delta g_{\mu\nu}$$

of reference metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

with

$$f(r) = 1 - \frac{\Lambda}{3}r^2.$$

Spliting $\delta g_{\mu\nu}$ into axial and polar classes, we'll find that eqs linearized gravity

$$\delta R_{\mu\nu} = \Lambda \delta g_{\mu\nu}$$

amount to Schrödinger problem

$$\left(-\frac{d^2}{dr_*^2} + f(r)\frac{\ell(\ell+1)}{r^2}\right)\Psi(r) = \omega^2\Psi(r),$$

which is same as for electromagnetism.

It is special case of perturbed Einstein-Maxwell system having $M = Q = 0$ (no sources).

For definiteness, consider perturbations of AdS_4 space-time, in which case

$$\tan\left(\sqrt{-\frac{\Lambda}{3}}\, r_*\right) = \sqrt{-\frac{\Lambda}{3}}\, r.$$

It is also convenient to use angle

$$x = \sqrt{-\frac{\Lambda}{3}}\, r_*$$

which runs $0 \ldots \pi/2$ as r runs $0 \ldots \infty$.

For dS_4 : $\tanh\left(\sqrt{\frac{\Lambda}{3}}\, r_*\right) = \sqrt{\frac{\Lambda}{3}}\, r.$

For Minkowski : $r_* = r.$

Axial perturbations

$$\delta g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} e^{-i\omega t} \sin\theta \, \partial_\theta P_\ell(\cos\theta)$$

satisfy linearized Einstein eqs provided that

- $h_0(x) = \frac{i}{\omega}\frac{d}{dx}\left(\tan x \; \Psi_{RW}(x)\right)$
- $h_1(x) = \sqrt{-\frac{3}{\Lambda}} \sin x \cos x \; \Psi_{RW}(x)$

where

$$\left(-\frac{d^2}{dx^2} + V_{RW}(x)\right)\Psi_{RW}(x) = \Omega^2 \Psi_{RW}(x)$$

with

$$\boxed{V_{RW}(x) = \frac{\ell(\ell+1)}{\sin^2 x}}$$

and

$$\Omega = \sqrt{-\frac{3}{\Lambda}}\,\omega .$$

Polar perturbations

$$\delta g_{\mu\nu} = \begin{pmatrix} f(r) H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & \frac{H_2(r)}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 K(r) & 0 \\ 0 & 0 & 0 & r^2 K(r) \sin^2\theta \end{pmatrix} e^{-i\omega t} P_\ell(\cos\theta)$$

satisfy linearized Einstein eqs provided that

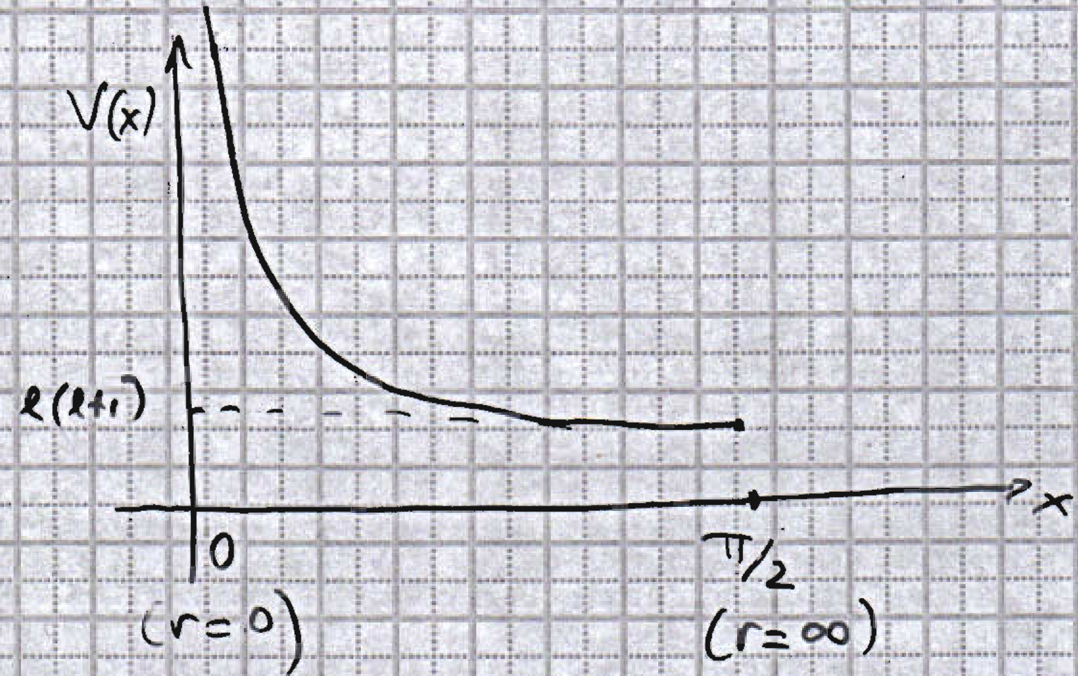
- $H_0(r) = H_2(r) = \sqrt{-\frac{\Lambda}{3}} \left(\frac{\ell(\ell+1)}{2} \cot x + \frac{3\omega^2}{\Lambda} \sin x \cos x + \cos^2 x \frac{d}{dx} \right) \Psi_Z(x)$
- $H_1(x) = -i\omega \cos x \frac{d}{dx} \left(\sin x \, \Psi_Z(x) \right)$
- $K(x) = \sqrt{-\frac{\Lambda}{3}} \left(\frac{\ell(\ell+1)}{2} \cot x + \frac{d}{dx} \right) \Psi_Z(x)$

where $\left(-\frac{d^2}{dx^2} + V_Z(x) \right) \Psi_Z(x) = \Omega^2 \Psi_Z(x)$

with $\boxed{V_Z(x) = \frac{\ell(\ell+1)}{\sin^2 x}}$

and $\Omega = \sqrt{-\frac{3}{\Lambda}}\, \omega$, as before

In both cases, the effective Schrödinger problem is



and can be transformed into hypergeometric equation

$$z(1-z)\frac{d^2Y}{dz^2}+\left[c-(a+b+1)z\right]\frac{dY}{dz}-abY=0$$

with coefficients

$$a=\frac{1}{2}(l+2+\Omega),\quad b=\frac{1}{2}(l+2-\Omega),\quad c=l+\frac{3}{2}$$

using the change of variables

$$z=\sin^2 x,\quad \Psi(x)=\cos x\sin^{l+1}x\;Y(z)$$

Thus, the normalizable solution $\Psi(0)=0$ takes the form

$$\Psi(x)=\cos x\sin^{l+1}x\;F(a,b;c;\sin^2 x)$$

in terms of hypergeometric functions.

$\mathcal{E}_{ab}$ and B_{ab} fields for axial perturbations

- $$\mathcal{E}_{\theta\phi} = -\frac{\cos^3 x}{2}\frac{d}{dx}\left(\sin x\, \Psi_{RW}^{(x)}\right) e^{-i\omega t} \sin\theta \left[\ell(\ell+1) + 2\cot\theta\, \partial_\theta\right] P_\ell$$

- $$\mathcal{E}_{t\phi} = \frac{i\Lambda}{6\omega}(\ell-1)(\ell+2)\frac{\cos^3 x}{\sin x}\left(\frac{d}{dx}\Psi_{RW}^{(x)}\right) e^{-i\omega t} \sin\theta\, \partial_\theta P_\ell$$

- $$B_{tt} = \frac{i}{2\omega}\left(\sqrt{-\frac{\Lambda}{3}}\right)^3 (\ell-1)\ell(\ell+1)(\ell+2)\frac{\cos^3 x}{\sin^3 x}\Psi_{RW}^{(x)} e^{-i\omega t} P_\ell$$

- $$B_{t\theta} = -\frac{1}{2}\sqrt{-\frac{\Lambda}{3}}(\ell-1)(\ell+2)\frac{\cos^3 x}{\sin x}\Psi_{RW}(x)\, e^{-i\omega t}\partial_\theta P_\ell$$

- $$\begin{aligned} B_{\substack{\theta\theta\\ \phi\phi}} = &\mp \frac{i}{2\omega}\sqrt{-\frac{\Lambda}{3}}\,\ell(\ell+1)\frac{\cos^3 x}{\sin x}\left[\left(\ell^2+\ell\pm 1+\frac{3\omega^2}{\Lambda}\sin^2 x\right)\Psi_{RW}^{(x)}\right.\\ &\left. + \sin x\cos x\frac{d}{dx}\Psi_{RW}(x)\right]\sin^2\theta\, e^{-i\omega t} P_\ell \\ &\mp \frac{i}{\omega}\sqrt{-\frac{\Lambda}{3}}\frac{\cos^3 x}{\sin x}\left[\left(\frac{\ell(\ell+1)}{2}+\frac{3\omega^2}{\Lambda}\sin^2 x\right)\Psi_{RW}(x)\right.\\ &\left. + \sin x\cos x\frac{d}{dx}\Psi_{RW}(x)\right] e^{-i\omega t}\underbrace{\cot\theta\, \partial_\theta P_\ell}_{\sin^2\theta} \end{aligned}$$

and all other components vanish.

$\mathcal{E}_{ab}$ and B_{ab} fields for polar perturbations

- $$\mathcal{E}_{tt} = -\frac{1}{4}\left(\sqrt{-\frac{\Lambda}{3}}\right)^3 (\ell-1)\ell(\ell+1)(\ell+2)\,\frac{\cos^3 x}{\sin^3 x}\,\Psi_z(x)\,e^{-i\omega t}\,P_\ell$$

- $$\mathcal{E}_{t\theta} = -\frac{i\omega}{4}\sqrt{-\frac{\Lambda}{3}}\,(\ell-1)(\ell+2)\,\frac{\cos^3 x}{\sin x}\,\Psi_z(x)\,e^{-i\omega t}\,\partial_\theta P_\ell$$

- $$\mathcal{E}_{\substack{\theta\theta\\ \phi\phi}} = \pm\frac{1}{4}\sqrt{-\frac{\Lambda}{3}}\,\ell(\ell+1)\,\frac{\cos^3 x}{\sin x}\Big[\Big(\ell^2+\ell\pm 1+\frac{3\omega^2}{\Lambda}\sin^2 x\Big)\Psi_z(x)$$
$$+\sin x\cos x\,\frac{d}{dx}\Psi_z(x)\Big]\sin^2\theta\, e^{-i\omega t}\,P_\ell$$
$$\pm\frac{1}{2}\sqrt{-\frac{\Lambda}{3}}\,\frac{\cos^3 x}{\sin x}\Big[\Big(\frac{\ell(\ell+1)}{2}+\frac{3\omega^2}{\Lambda}\sin^2 x\Big)\Psi_z(x)$$
$$+\sin x\cos x\,\frac{d}{dx}\Psi_z(x)\Big]e^{-i\omega t}\underset{\sin^2\theta}{\wedge}\cot\theta\,\partial_\theta P_\ell$$

- $$B_{\theta\phi} = \frac{i\omega}{4}\cos^3 x\,\frac{d}{dx}\big(\sin x\,\Psi_z(x)\big)\,e^{-i\omega t}\sin\theta\big[\ell(\ell+1)+2\cot\theta\,\partial_\theta\big]P_\ell$$

- $$B_{t\phi} = \frac{\Lambda}{12}(\ell-1)(\ell+2)\,\frac{\cos^3 x}{\sin x}\Big(\frac{d}{dx}\Psi_z(x)\Big)e^{-i\omega t}\sin\theta\,\partial_\theta P_\ell$$

and all other components vanish, as before.

Close inspection reveals

$$\boxed{\begin{aligned} \mathcal{E}^{\text{polar}}_{ab} &= B^{\text{axial}}_{ab} \,, \\ B^{\text{polar}}_{ab} &= -\,\mathcal{E}^{\text{axial}}_{ab} \,, \end{aligned}}$$

provided that

$$\Psi_{RW}(x) = \frac{i\omega}{2}\,\Phi_{Z}(x) \,.$$

i.e., when axial and polar perturbations satisfy the same boundary conditions at $r=\infty$ so that the ω's are the same.

Electric-Magnetic duality of linearized gravity in a nut-shell reads

$$\boxed{\text{axial} \rightleftarrows \text{polar}}$$

- Although derived for perturbations of AdS_4 space-time, the same result holds for perturbations around flat $(\Lambda=0)$ and dS_4 $(\Lambda>0)$ space-times

Holographic manifestation

We compute the energy-momentum tensor for perturbations of AdS_4 space-time by holographic renormalization:

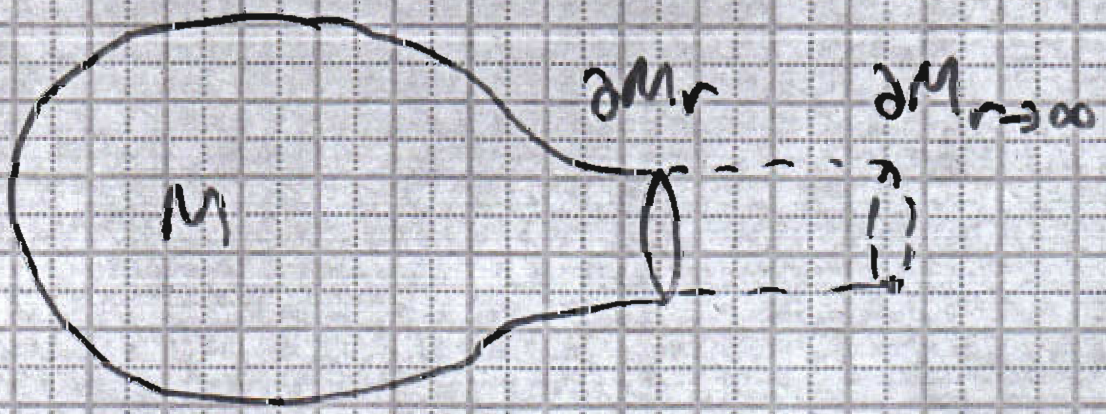
$$T^{ab} = \frac{2}{\sqrt{-\det\gamma}} \frac{\delta S'_{grav}}{\delta \gamma_{ab}}$$

where

$$S'_{grav} = \frac{1}{2K^2} \int_M d^4x \sqrt{-\det g}\,\left(R[g] - 2\Lambda\right) - \frac{1}{K^2} \int_{\partial M} d^3x \sqrt{-\det\gamma}\, K - \frac{2}{K^2}\sqrt{-\frac{\Lambda}{3}} \int_{\partial M} d^3x \sqrt{-\det\gamma}\left(1 - \frac{3}{4\Lambda} R[\gamma]\right)$$

and obtain

$$K^2 T_{ab} = K_{ab} - K\gamma_{ab} - 2\sqrt{-\frac{\Lambda}{3}}\,\gamma_{ab} + \sqrt{-\frac{3}{\Lambda}}\, G_{ab}[\gamma]$$



The boundary metric is

$$ds_{\mathcal{J}}^2 = \lim_{r\to\infty}\left(-\frac{3}{\Lambda r^2}\gamma_{ab}\,dx^a dx^b\right)$$

and the renormalized energy-momentum tensor

$$T_{ab}^{\text{renorm}} = \lim_{r\to\infty}\left(\sqrt{-\frac{\Lambda}{3}}\,r\,T_{ab}\right)$$

is finite, traceless and conserved.

Use asymptotic expansion, as $r\to\infty$, of effective wave-function

$$\Phi_{RW}(r) = I_0 + \frac{I_1}{r} + \frac{I_2}{r^2} + \frac{I_3}{r^3} + \cdots$$

where

$$I_0 = \Gamma^{-1}\left(\tfrac{1}{2}(\ell+2+\Omega)\right)\Gamma^{-1}\left(\tfrac{1}{2}(\ell+2-\Omega)\right)$$

$$I_1 = -2\sqrt{-\frac{3}{\Lambda}}\,\Gamma^{-1}\left(\tfrac{1}{2}(\ell+1+\Omega)\right)\Gamma^{-1}\left(\tfrac{1}{2}(\ell+1-\Omega)\right)$$

and impose general boundary conditions

$$\gamma = \frac{I_0}{I_1} \equiv -\sqrt{-\frac{\Lambda}{3}}\,\frac{\Phi(x)}{d\Phi/dx}\bigg|_{r=\infty}$$

The spectrum $\Omega = \sqrt{-\frac{3}{\Lambda}}\,\omega$ depends upon γ.

Axial perturbations:

The boundary metric on $\mathcal{I}$ takes the form

$$ds^2_{\mathcal{I}} = -dt^2 - \frac{3}{\Lambda}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) + $$
$$+ 2\frac{i I_0}{\omega} e^{-i\omega t} \sin\theta\, \partial_\theta P_\ell \cdot dt\, d\varphi$$

and the non-vanishing components of the energy-momentum tensor are

- $\kappa^2 T_{\theta\varphi} = -\frac{I_1}{2} e^{-i\omega t} \sin\theta \left[\ell(\ell+1) + 2\cot\theta\, \partial_\theta\right] P_\ell$
- $\kappa^2 T_{t\varphi} = \frac{i\Lambda}{6\omega} (\ell-1)(\ell+2) I_1\, e^{-i\omega t} \sin\theta\, \partial_\theta P_\ell$

- $ds^2_{\mathcal{I}}$ is static (and conformally flat) only when Ψ satisfies Dirichlet bound. cond ($I_0 = 0$)
- T_{ab} vanishes only when Ψ satisfies Neumann boundary conditions ($I_1 \neq 0$)
- In general they both vary with time

Polar perturbations

Using similar expansion for Ψ_Z wave-function

$$\Psi_Z(r) = J_0 + \frac{J_1}{r} + \frac{J_2}{r^2} + \frac{J_3}{r^3} + \cdots$$

we obtain the boundary metric

$$ds_J^2 = -dt^2 - \frac{3}{\Lambda}\left[1 + \frac{\Lambda}{3} J_1 e^{-i\omega t} P_\ell\right](d\theta^2 + \sin^2\theta\, d\phi^2)$$

and the non-vanishing compts of T_{ab} are

$$\kappa^2 T_{tt} = -\frac{\Lambda}{12}(\ell-1)\ell(\ell+1)(\ell+2) J_0 e^{-i\omega t} P_\ell$$

$$\kappa^2 T_{t\theta} = \frac{i\omega}{4}(\ell-1)(\ell+2) J_0 e^{-i\omega t} \partial_\theta P_\ell$$

$$\kappa^2 T_{\substack{\theta\theta \\ \phi\phi}} = \mp \frac{1}{4}\ell(\ell+1)\left(\ell(\ell+1) \pm 1 + \frac{3\omega^2}{\Lambda}\right) J_0 \underbrace{e^{-i\omega t} P_\ell}_{\sin^2\theta}$$

$$\mp \frac{1}{4}\left(\ell(\ell+1) + \frac{6\omega^2}{\Lambda}\right) J_0 \underbrace{e^{-i\omega t}}_{\sin^2\theta} \cot\theta\, \partial_\theta P_\ell$$

- ds_J^2 is static (and conformally flat) only when Ψ satisfies Neumann bound. cond. ($J_1 = 0$)
- T_{ab} vanishes only when Ψ satisfies Dirichlet bound. cond. ($J_0 = 0$)
- In general they both vary with time

Cotton tensor

In 3-d the deviations from conformally flat metric are measured by Cotton tensor

$$C^{ab} = \frac{1}{2\sqrt{-\det\gamma}} \left(\varepsilon^{acd} \nabla_c R^b{}_d + \varepsilon^{bcd} \nabla_c R^a{}_d \right)$$

$$= \frac{1}{2\sqrt{-\det\gamma}} \frac{\delta S_{CS}}{\delta\gamma^{ab}}$$

using the gravitational CS action

$$S_{CS} = \int d^3x \sqrt{-\det\gamma}\, \varepsilon^{abc} \Gamma^d_{ae} \left(\partial_b \Gamma^e_{cd} + \frac{2}{3} \Gamma^e_{bf} \Gamma^f_{cd} \right)$$

<u>For polar perturbations</u>, the boundary metric has

- $C_{\theta\phi} = \frac{\Lambda}{12} i\omega J_1 e^{-i\omega t} \sin\theta \left[\ell(\ell+1) + 2\cot\theta\, \partial_\theta \right] P_\ell$
- $C_{t\phi} = \frac{\Lambda^2}{36} (\ell-1)(\ell+2) J_1 e^{-i\omega t} \sin\theta\, \partial_\theta P_\ell$

and so choosing $I_1 = -\frac{\Lambda}{6} i\omega J_1$, it follows

$$\boxed{C_{ab}\,(\text{polar}) = \kappa^2\, T_{ab}\,(\text{axial})}$$

under same boundary conditions at $r=\infty$.

For axial perturbations, the boundary metric be

- $$C_{tt} = \frac{i\Lambda^2}{18\omega}(\ell-1)\ell(\ell+1)(\ell+2)\, I_0\, e^{-i\omega t} P_\ell$$

- $$C_{t\theta} = \frac{\Lambda}{6}(\ell-1)(\ell+2)\, I_0\, e^{-i\omega t}\, \partial_\theta P_\ell$$

- $$C_{\substack{\theta\theta \\ \phi\phi}} = \pm \frac{i\Lambda}{6\omega}\ell(\ell+1)\left(\ell(\ell+1) \pm 1 + \frac{3\omega^2}{\Lambda}\right) I_0\, e^{-i\omega t} \underset{\sin^2\theta}{P_\ell}$$
$$\pm \frac{i\Lambda}{6\omega}\left(\ell(\ell+1) + \frac{6\omega^2}{\Lambda}\right) I_0\, e^{-i\omega t} \underset{\sin^2\theta}{\cot\theta\, \partial_\theta P_\ell}$$

and so choosing $J_0 = -i\frac{2\Lambda}{3\omega} I_0$, it follows

$$\boxed{C_{ab}(\text{axial}) = k^2\, T_{ab}(\text{polar})}$$

under same boundary conditions at $r=\infty$.

Thus axial $\rightleftarrows$ polar manifests as

$C_{ab} \rightleftarrows T_{ab}$ at the boundary of AdS_4.

It can be regarded as holographic manifestation of electric-magnetic gravitational duality of bulk theory.

Actually, it turns out that

$$\lim_{r\to\infty}\left(\frac{\Lambda}{3} r^3 \mathcal{E}_{ab}\right) = \kappa^2 T_{ab}$$

$$\lim_{r\to\infty}\left(\frac{\Lambda^2}{9} r^3 B_{ab}\right) = C_{ab}$$

for either type of perturbation and so:

Electric/Magnetic duality on the bulk manifests as

Energy-momentum/Cotton tensor duality on the boundary,

which is simply realized by interchange of axial and polar perturbations satisfying the same boundary conditions.

This is also known as dual graviton correspondence.