

# Strings in curved space-time

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- The features of the space-time geometry (connection, torsion and nonmetricity)
  - B. Sazdović
    - \* Torsion and nonmetricity in the stringy geometry (*hep-th/0304086*)
    - \* Bosonic string theory in background fields by canonical methods *IJMP A 20* (2005) 5501;
  - D. S. Popović and B. Sazdović  
The geometrical form for the string space-time action  
*Eur. Phys. J.* (2007) (*hep-th/ 0701264*)
- Decreasing the number of the Dp-brane dimensions
  - B. Nikolić and B. Sazdović
    - \* Gauge symmetries decrease the number of Dp-brane dimensions  
*Phys. Rev. D 74* (2006) 045024 (*hep-th/ 0604129*)
    - \* Gauge symmetries decrease the number of Dp-brane dimensions, II (Inclusion of the Liouville term) (*hep-th/ 0611191*)
- Symmetries of the space-time field equations

## Definition of the model

- Action

$$S = \kappa \int_{\Sigma} d^2\xi \sqrt{-g} \left\{ \left[ \frac{1}{2} g^{\alpha\beta} G_{\mu\nu} + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu} \right] \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu} + \Phi R^{(2)} \right\}$$

- $\xi^{\alpha}$  ( $\alpha = 0, 1$ ) world-sheet coordinates
- $x^{\mu}(\xi)$  ( $\mu = 0, 1, \dots, D - 1$ ) space-time coordinates
- $x^i(\xi)$  ( $i = 0, 1, \dots, p$ ) Dp-brane coordinates
- String propagates in background defined by
  - \* metric tensor  $G_{\mu\nu}(x)$
  - \* antisymmetric tensor field  $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$
  - \* dilaton field  $\Phi(x)$

- Space-time field equations (Consequence of the quantum world-sheet conformal invariance)

$$\beta_{\mu\nu}^G \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} + 2D_{\mu} a_{\nu} = 0$$

$$\beta_{\mu\nu}^B \equiv D_{\rho} B^{\rho}{}_{\mu\nu} - 2a_{\rho} B^{\rho}{}_{\mu\nu} = 0$$

$$\beta^{\Phi} \equiv 2\pi\kappa \frac{D-26}{6} - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_{\mu} a^{\mu} + 4a^2 = 0$$

$$B_{\mu\rho\sigma} = \partial_{\mu} B_{\nu\rho} + \partial_{\nu} B_{\rho\mu} + \partial_{\rho} B_{\mu\nu} \quad \text{field strength}$$

$$a_{\mu} = \partial_{\mu} \Phi \quad \text{gradient of dilaton}$$

## Canonical analysis

- Currents

$$J_{\pm\mu} = P^T{}_{\mu}{}^{\nu} j_{\pm\nu} + \frac{a_{\mu}}{2a^2} i_{\pm}^{\Phi} = j_{\pm\mu} - \frac{a_{\mu}}{a^2} j$$

$$i_{\pm}^F = \frac{a^{\mu}}{a^2} j_{\pm\mu} - \frac{1}{2a^2} i_{\pm}^{\Phi} \pm 2\kappa F', \quad i_{\pm}^{\Phi} = \pi_F \pm 2\kappa \Phi'$$

where

$$j_{\pm\mu} = \pi_{\mu} + 2\kappa \Pi_{\pm\mu\nu} x^{\nu'}, \quad \Pi_{\pm\mu\nu} \equiv B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$$

$$j = a^{\mu} j_{\pm\mu} - \frac{1}{2} i_{\pm}^{\Phi} = a^2 (i_{\pm}^F \mp 2\kappa F')$$

$\pi_{\mu}$  and  $\pi_F$  are canonically conjugate momenta to the variables  $x^{\mu}$  and  $F$

- Canonical Hamiltonian and energy momentum tensor

$$\mathcal{H}_c = h^- T_- + h^+ T_+$$

$$T_{\pm} = \mp \frac{1}{4\kappa} \left( G^{\mu\nu} J_{\pm\mu} J_{\pm\nu} + i_{\pm}^F i_{\pm}^{\Phi} \right) + \frac{1}{2} i_{\pm}^{\Phi'}$$

$$= \mp \frac{1}{4\kappa} \left( G^{\mu\nu} j_{\pm\mu} j_{\pm\nu} - \frac{j^2}{a^2} \right) + \frac{1}{2} (i_{\pm}^{\Phi'} - F' i_{\pm}^{\Phi})$$

- Two independent copies of Virasoro algebras

$$\{T_{\pm}(\sigma), T_{\pm}(\bar{\sigma})\} = -[T_{\pm}(\sigma) + T_{\pm}(\bar{\sigma})] \delta'(\sigma - \bar{\sigma})$$

## Equations of motion

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$$[J^\mu] \equiv \nabla_{\mp} \partial_{\pm} x^\mu + {}^* \Gamma_{\mp \rho \sigma}^\mu \partial_{\pm} x^\rho \partial_{\mp} x^\sigma = 0$$

$$[h^\pm] \equiv G_{\mu\nu} \partial_{\pm} x^\mu \partial_{\pm} x^\nu - 2 \nabla_{\pm} \partial_{\pm} \Phi = 0$$

$$[i^F] \equiv R^{(2)} + \frac{2}{a^2} (D_{\mp \mu} a_\nu) \partial_{\pm} x^\nu \partial_{\mp} x^\mu = 0$$

- Generalized connection

$${}^* \Gamma_{\pm \nu \mu}^\rho = \Gamma_{\pm \nu \mu}^\rho + \frac{a^\rho}{a^2} D_{\pm \mu} a_\nu = \Gamma_{\nu \mu}^\rho \pm P^{T\rho}{}_\sigma B_{\nu \mu}^\sigma + \frac{a^\rho}{a^2} D_{\mu} a_\nu$$

Under space-time general coordinate transformations the expression  ${}^* \Gamma_{\pm \nu \mu}^\rho$  transforms as a connection.

- The covariant derivatives with respect to the Christoffel connection  $\Gamma_{\nu \mu}^\rho$  and to the connection  $\Gamma_{\pm \nu \mu}^\rho = \Gamma_{\nu \mu}^\rho \pm B_{\nu \mu}^\rho$ , we respectively denote as  $D_\mu$  and  $D_{\pm \mu}$

## Geometry of space-time seen by the probe string

- Parallel transport

$$V^\mu(x) \rightarrow {}^\circ V_{\parallel}^\mu(x + dx) = V^\mu + {}^\circ \delta V^\mu$$

$${}^\circ \delta V^\mu = -{}^\circ \Gamma_{\rho\sigma}^\mu V^\rho dx^\sigma$$

${}^\circ \Gamma_{\rho\sigma}^\mu$  affine linear connection

- The covariant derivative

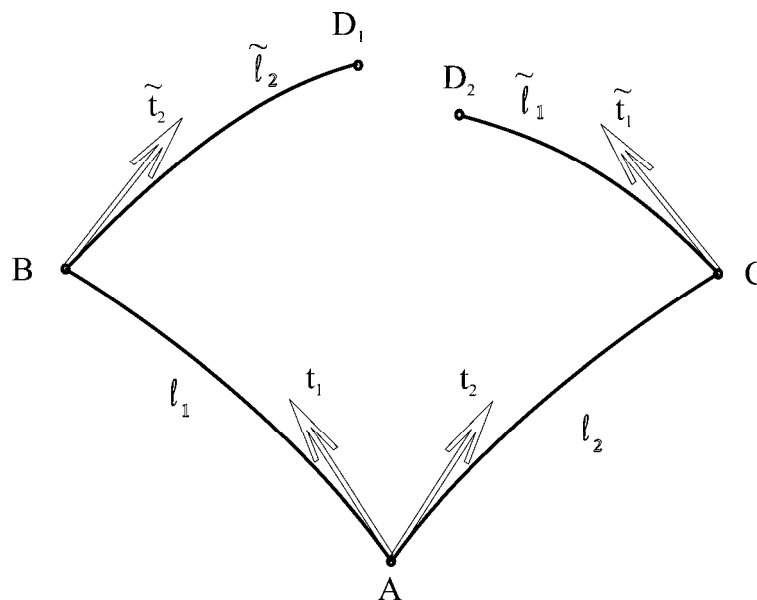
$$\begin{aligned} {}^\circ DV^\mu &= V^\mu(x + dx) - {}^\circ V_{\parallel}^\mu = dV^\mu - {}^\circ \delta V^\mu \\ &= (\partial_\nu V^\mu + {}^\circ \Gamma_{\rho\nu}^\mu V^\rho) dx^\nu \equiv {}^\circ D_\nu V^\mu dx^\nu \end{aligned}$$

## Torsion

- **torsion** (antisymmetric part of the affine connection)

$${}^{\circ}T^{\rho}_{\mu\nu} = {}^{\circ}\Gamma^{\rho}_{\mu\nu} - {}^{\circ}\Gamma^{\rho}_{\nu\mu}$$

Geometrical interpretation: measures the non-closure of the curved "parallelogram"



$$x^{\mu}(D_2) - x^{\mu}(D_1) = {}^{\circ}T^{\mu}_{\rho\sigma} t_1^{\rho} t_2^{\sigma} dl_1 dl_2$$

## Non-metricity

- Covariant derivative is responsible for the comparison of the vectors from different points
- What variable is responsible for comparison of the lengths of the vectors?

- $V^\mu(x)$ : using invariance of the scalar product under the parallel transport

$$\begin{aligned} V^2(x) &= G_{\mu\nu}(x)V^\mu(x)V^\nu(x) \\ &= [G_{\mu\nu}(x) + {}^\circ\delta G_{\mu\nu}(x)] {}^\circ V_{\parallel}^\mu {}^\circ V_{\parallel}^\nu \end{aligned}$$

- its parallel transport to the point  $x + dx$ ,  ${}^\circ V_{\parallel}^\mu$ ,

$${}^\circ V_{\parallel}^2(x + dx) = G_{\mu\nu}(x + dx) {}^\circ V_{\parallel}^\mu {}^\circ V_{\parallel}^\nu$$

- Difference of the squares of the vectors

$$\begin{aligned} {}^\circ\delta V^2 &= {}^\circ V_{\parallel}^2(x + dx) - V^2(x) \\ &= [G_{\mu\nu}(x + dx) - G_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)] {}^\circ V_{\parallel}^\mu {}^\circ V_{\parallel}^\nu \end{aligned}$$

Up to the higher order terms we have

$$\begin{aligned} {}^\circ\delta V^2 &= [dG_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)] V^\mu V^\nu \\ &= {}^\circ D G_{\mu\nu} V^\mu V^\nu \equiv -dx^\rho {}^\circ Q_{\rho\mu\nu} V^\mu V^\nu \end{aligned}$$

- **nonmetricity** covariant derivative of the metric tensor

$${}^\circ Q_{\mu\rho\sigma} = -{}^\circ D_\mu G_{\rho\sigma}$$

## Stringy torsion and nonmetricity

- **Stringy torsion** antisymmetric part of the stringy connection

$${}^*T_{\pm\mu\nu}{}^\rho = {}^*\Gamma_{\pm\mu\nu}{}^\rho - {}^*\Gamma_{\pm\nu\mu}{}^\rho = \pm 2P^{T\rho}{}_\sigma B_{\mu\nu}^\sigma$$

- **Stringy nonmetricity** measures non-compatibility of the metric  $G_{\mu\nu}$  with the stringy connection  ${}^*\Gamma_{\pm\nu\rho}{}^\mu$

$${}^*Q_{\pm\mu\rho\sigma} \equiv -{}^*D_{\pm\mu}G_{\rho\sigma} = \frac{1}{a^2}D_{\pm\mu}(a_\rho a_\sigma)$$

The presence of the dilaton field  $\Phi$  leads to breaking of the space-time metric postulate

- **Stringy Weyl vector**

$${}^*q_\mu = \frac{1}{D}G^{\rho\sigma}{}^*Q_{\pm\mu\rho\sigma} = \frac{-4}{D}\partial_\mu\varphi$$

is a gradient of new scalar field  $\varphi$ , defined by the expression

$$\varphi = -\frac{1}{4}\ln a^2 = -\frac{1}{4}\ln(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi)$$



## The space-time action

$$S = \int dx \sqrt{-G} e^{-2\Phi} \left[ R - \frac{1}{12} B^2 + 2D^2\Phi \right]$$

$$B^2 = B_{\mu\nu\rho} B^{\mu\nu\rho} \quad D^2\Phi = G^{\mu\nu} D_\mu \partial_\nu \Phi$$

is reproduced by the action

$${}^*S = \int d^D x \sqrt{-G} e^{-2\varphi} {}^*\mathcal{L}$$

$${}^*\mathcal{L} \equiv {}^*R + \frac{1}{48} \left( 11 {}^*T^2 - 26 {}^*Q^2 \right) + \frac{1}{3} \left( \frac{5D}{4} \right) {}^*q^2$$

If the condition  $\varphi = \Phi \iff G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - e^{-4\Phi} = 0$  is satisfied, up to the term with factor  $\frac{1}{a^2}$   $S = {}^*S$

Invariant measure

1. Invariant under space-time general coordinate transformations.

2. Preserved under parallel transport  $\iff {}^*D_{\pm\mu} {}^*\Omega = 0$ .

3. Enable integration by parts, which can be achieved with help of the Leibniz rule and the relation

$$\int d^D x {}^*\Omega {}^*D_{\pm\mu} V^\mu = \int d^D x \partial_\mu ({}^*\Omega V^\mu)$$

so that we are able to use Stoke's theorem.

## Decreasing the number of the Dp-brane dimensions

### Open string boundary conditions

- According to the **action principle**, evolution from given initial configuration (at  $\tau_i$ ) to given final configuration (at  $\tau_f$ ) is such that the action is stationary

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left( \dot{p}_\mu + \gamma'_\mu \right) \delta x^\mu + \int_{\tau_i}^{\tau_f} d\tau (\gamma_\mu \delta x^\mu) \Big|_{\sigma=0}^{\sigma=\pi} = 0$$

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \quad \gamma_\mu \equiv \frac{\partial \mathcal{L}}{\partial x'^\mu}$$

- For all variations  $\delta x^\mu \rightarrow$  EM  $\dot{p}_\mu + \gamma'_\mu = 0$
- At the string endpoints  $\rightarrow$  BC  $(\gamma_\mu \delta x^\mu) \Big|_{\sigma=0}^{\sigma=\pi} = 0$

- Closed string: do not have endpoints  $\rightarrow$  no BC

- Open string

- *arbitrary variations  $\delta x^\mu$  on the string endpoints*  
 $\rightarrow$  **Neumann** boundary conditions

$$\gamma_\mu \Big|_{\sigma=0} = 0, \quad \gamma_\mu \Big|_{\sigma=\pi} = 0$$

- *fixed edges of the string*  
 $\rightarrow$  **Dirichlet** boundary conditions

$$\delta x^\mu \Big|_{\sigma=0} = 0, \quad \delta x^\mu \Big|_{\sigma=\pi} = 0$$

## Dp-brane

- –  $x^i = \{x^0, x^1, \dots, x^p\}$  *Dp tangential coordinates*  
**NN** Boundary conditions:  $\gamma_i / \sigma = \sigma^* = 0$
  - $x^a = \{x^{p+1}, x^{p+2}, \dots, x^{D-1}\}$  *Dp normal coordinates*  
**DD** Boundary conditions:  $\delta x^i |_{\sigma^*} = 0$
- Example:  $D = 3$   
 $x^i = \{x^0 \equiv X, x^1 \equiv Z\}$ ,  $x^a \equiv \{x^3 = Y = 0\}$

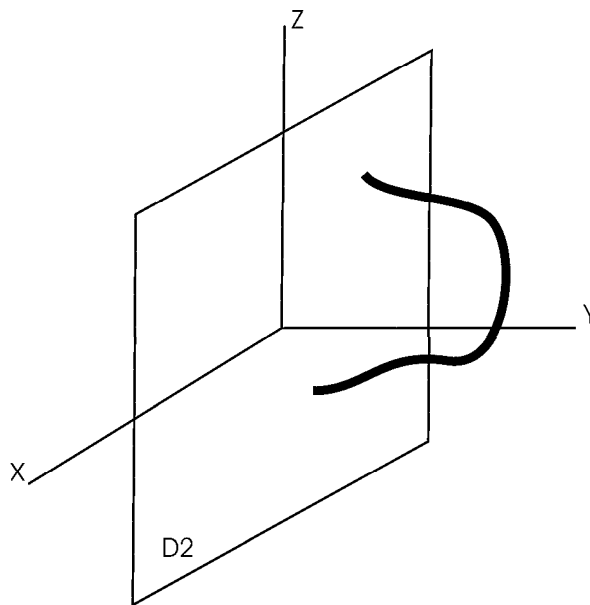


Figure 1: D2-brane stretched over (X,Z) plane

- String endpoints are attached to Dp-brane and move freely on  $(x^0, x^1)$  plane

## Linear dilaton background

- – Open string propagates in **linear dilaton background**
  - \* metric tensor  $G_{\mu\nu} = \text{const}$
  - \* antisymmetric tensor field  $B_{\mu\nu} = -B_{\nu\mu} = \text{const}$
  - \* dilaton field  $\Phi = \Phi_0 + a_\mu x^\mu$ , ( $a_\mu = \text{const}$ )
- Chose for simplicity
  - \*  $B_{\mu\nu} \rightarrow B_{ij}$  and  $a_\mu \rightarrow a_i$
  - \*  $G_{\mu\nu} = 0$  for  $\mu = i \in \{0, 1, \dots, p\}$   
 $\nu = a \in \{p+1, \dots, D-1\}$
- Conformal gauge,  $g_{\alpha\beta} = e^{2F} \eta_{\alpha\beta}$

$$S = \kappa \int_{\Sigma} d^2\xi \left\{ \left[ \frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + 2\eta^{\alpha\beta} a_i \partial_\alpha x^i \partial_\beta F \right\}$$

Action depends on  $F \longrightarrow$  no conformal invariance

## Non-commutativity in absence of diaton field,

$$\Phi = 0$$

- For  $\Phi = 0$ , action is  $F$  independent
- – Consider  $\gamma_i|_{\sigma=\sigma^*}$  as primary constraints
- Apply Dirac consistency procedure  $\gamma_i|_{\sigma=\sigma^*} \rightarrow \Gamma_i(\sigma)$
- – Algebra of constraints

$$\{\Gamma_i(\sigma), \Gamma_j(\bar{\sigma})\} = -\kappa G_{ij}^{eff} \delta'(\sigma - \bar{\sigma})$$

$G_{ij}^{eff} \equiv G_{ij} - 4B_{ik}G^{kq}B_{qj}$  **effective metric tensor**

- We will refer to it as the **open string metric tensor**, the metric tensor seen by the open string

- Conventions:  $(G_{eff}^{-1})^{ij}$  inverse of  $G_{ij}^{eff}$
- $\tilde{V}^i = (G_{eff}^{-1})^{ij} V_j$        $\tilde{V}^2 = (G_{eff}^{-1})^{ij} V_i V_j$
- $V^i = G^{ij} V_j$        $V^2 = G^{ij} V_i V_j$

- – We solve SCC constraints  $\Gamma_i = 0$

$$x^i(\sigma) = q^i(\sigma) - 2\Theta^{ij} \int_0^\sigma d\sigma_1 p_j(\sigma_1)$$

$$q^i(\sigma) = \frac{1}{2} [x^i(\sigma) + x^i(-\sigma)] , \quad p_i(\sigma) = \frac{1}{2} [\pi_i(\sigma) + \pi_i(-\sigma)]$$

- All Dp-brane coordinates are non-commutative

$$\{x^i(\sigma), x^j(\bar{\sigma})\} = 2\Theta^{ij} \theta(\sigma + \bar{\sigma}), \quad \Theta^{ij} = -\frac{1}{\kappa} (G_{eff}^{-1} B G^{-1})^{ij}$$

## Linear dilaton and role of field $F$ in open string theory

- – In linear dilaton background  $\Phi = \Phi_0 + a_i x^i$ , classical theory is  $F$  dependent
  - Closed string theory: on the quantum level the field  $F$  decouples (*conformal invariance condition*)
  - Open string theory: *Whether field  $F$  decouples?*
- – Conformal part of the world-sheet metric  $F \rightarrow$  dynamical variable
  - $F \rightarrow$  **Neumann** boundary conditions  
*arbitrary variations  $\delta F$  on the string endpoints*

$$\gamma|_{\sigma=0} = 0, \quad \gamma|_{\sigma=\pi} = 0 \quad \left( \gamma \equiv \frac{\partial \mathcal{L}}{\partial F'} \right)$$

- Consider  $\gamma_i|_{\sigma=\sigma^*}$  and  $\gamma|_{\sigma=\sigma^*}$  as primary constraints
- Dirac consistency procedure  
 $\gamma_i|_{\sigma=\sigma^*} \rightarrow \Gamma_i(\sigma), \quad \gamma|_{\sigma=\sigma^*} \rightarrow \Gamma(\sigma)$
- 1.  $a^2 \neq 0$  and  $\tilde{a}^2 \neq 0$ 
  - turns coordinate  $x_c = a_i x^i$  to **commutative** one
  - $F$  becomes a **noncommutative** variable
  - Field  $F$  breaks conformal invariance of open string theory  
Sazdović, Eur. Phys. J. C44 (2005) 599, *hep-th/0408131*

## Origins of gauge symmetries

- 2.  $a^2 \equiv G^{ij}a_i a_j = 0$  and  $\tilde{a}^2 \neq 0$

$$j \equiv a^i \pi_i - \frac{1}{2}\pi + 2\kappa a^i B_{ij} x^{j'} = 2\kappa a^2 \dot{F},$$

Standard canonical constraint

- 3.  $a^2 \neq 0$  and  $\tilde{a}^2 \equiv (G_{eff}^{-1})^{ij}a_i a_j = 0$ 
  - Complete set of the constraints  $\chi_A = (\Gamma_i, \Gamma)$

$$\{\chi_A(\sigma), \chi_B(\bar{\sigma})\} = -\kappa M_{AB} \delta',$$

$$M_{AB} = \begin{pmatrix} \tilde{G}_{ij} & 2a_i \\ 2a_j & 0 \end{pmatrix}.$$

$$\tilde{G}_{ij} = G_{ij} - 4B_{ik}(P_T^0)^{kl}B_{lj} = (\hat{P}_T^1 G_{eff})_{ij}$$

$$\det M_{AB} = -4\tilde{a}^2 \det \tilde{G}_{ij},$$

- Two zeros in  $\tilde{a}^2 = 0 \rightarrow$  two first class constraints
- Some of the constraints originated from the boundary conditions **turn from the second class into the first class constraints**

## Effective theory

- – First class constraints generate local symmetries
- After gauge fixing, the gauge conditions and the first class constraints can be treated as second class constraints
- We solve them together with the true second class constraints

- The original variables in terms of new ones

$$x_{Dp}^i(\sigma) = Q^i(\sigma) - 2\Theta^{ij} \int_0^\sigma d\sigma_1 P_j(\sigma_1),$$

$$(aB)_i x^i /_{\sigma=0,\pi} = 0, \quad F = 0$$

- – Effective variables:  $Q^i = (P_{Dp})^i_j q^j$ ,  $P_i = (P_{Dp})_i^j p_j$
- $Q^i$  and  $P_i$  are canonical variables in the Dp-brane subspace defined by projector  $P_{Dp}$

$$\{Q^i(\sigma), P_j(\bar{\sigma})\} = (P_{Dp})^i_j \delta_s(\sigma, \bar{\sigma})$$

- Noncommutativity tensor

$$\Theta^{ij} = -\frac{1}{\kappa} (G_{eff}^{-1} P_{Dp} B G^{-1} P_{Dp})^{ij} \quad (\Theta^{ij} = -\Theta^{ji})$$

- Projectors  $(\Pi_1)_i^j = \frac{n_i \tilde{n}^j}{\tilde{n}^2}$   $(n_1)_i = (aB)_i$   
 $(\Pi_T^1)_i^j = \delta_i^j - (\Pi_1)_i^j$   $(P_{Dp})^i_j = (\Pi_T^1)_i^j /_{a_*^2=0}$



## Solution of boundary conditions-General features

- – Components  $(x_1, F)$  satisfy Dirichlet boundary conditions
  - First class constraints turn some Neuman to Dirichlet boundary conditions and  
**number of Dp-brane dimensions decreases**
  
- – **Commutative** degrees of freedom  $x_c = a_i x_{Dp}^i$ 
  - $a_i \Theta^{ij} = 0$ , so the effective momentum disappears
  - *Closed string components contain only the open string coordinates, while the momenta are absent*
  
- **Noncommutativity**

*The closed string components  $(x_{nc})^i$ , contains both the open string coordinates and momenta*

  - Separate the center of mass  

$$x_{Dp}^i(\sigma) = (x_{Dp}^i)_{cm} + X_{Dp}^i(\sigma)$$
  - Poisson brackets between the coordinates

$$\{X_{Dp}^i(\tau, \sigma), X_{Dp}^j(\tau, \bar{\sigma})\} = \Theta^{ij} \begin{cases} -1 & \sigma = 0 = \bar{\sigma} \\ 1 & \sigma = \pi = \bar{\sigma} \\ 0 & \text{otherwise} \end{cases} .$$

## Conclusion

- Dp-brane features

	$f_{cc}$	$s_{cc}$	$D_{Dp}$	$(P_{Dp})$	DBC	$(x_{nc})^i$	$x_c$
1	0	p+2	p+2	$\delta_i^j$	-	$(\Pi_T^0 x)^i, F$	$x_0$
2	1	p+2	p	$(P_T^1)_i^j$	$x_1, F$	$(P_T x)^i$	$x_0$
3	2	p	p	$(\hat{P}_T^1)_i^j$	$x_1, F$	$(\hat{P}_T x)^i$	$x_0$

$$x_0 = a_i x^i, \quad x_1 = (aB)_i x^i$$

$$(P_T^1)_i^j = (\Pi_T^1)_i^j \Big|_{a^2=0}, \quad (\hat{P}_T^1)_i^j = (\Pi_T^1)_i^j \Big|_{\tilde{a}^2=0}$$

- In cases 2. and 3. field  $F$  disappears, ( $F = 0$ )
- Additional, open string conformally invariant conditions

$$a^2 = 0 \quad \text{or} \quad \tilde{a}^2 = 0$$

Nikolić and Sazdović

Gauge symmetries decrease the number of Dp-brane dimensions

*Phys. Rev. D* **74** (2006) 045024 (*hep-th/0604129*)

- In all cases

- effective energy-momentum tensor satisfies Virasoro algebra

$$\{T_{\pm}, T_{\pm}\} = - [T_{\pm}(\sigma) + T_{\pm}(\bar{\sigma})] \delta', \quad \{T_{\pm}, T_{\mp}\} = 0,$$

- but in **new background**

$$G_{ij} \rightarrow (P_{Dp} G_{eff})_{ij}, \quad B_{ij} \rightarrow 0, \quad \Phi \rightarrow \Phi_0 + a_i Q^i$$

## Conformal invariance with the help of Liouville action

- – New conditions for quantum conformal invariance  
 $\beta_{\mu\nu}^G = 0$  ,  $\beta_{\mu\nu}^B = 0 \implies \beta^\Phi = c$ .
  - Non-linear sigma model becomes conformal field theory
  - There exists a Virasoro algebra with central charge  $c$
- – The remaining anomaly, can be cancelled by introducing Liouville action

$$S_L = -\frac{\beta^\Phi}{2(4\pi)^2\kappa} \int_\Sigma d^2\xi \sqrt{-g} R^{(2)} \frac{1}{\Delta} R^{(2)}$$

- Advantages
  - \* Conformal invariance in the presence of field  $F$
  - \* Noncommutativity parameter depend on central charge  $c$

## Realization

- Conformal gauge  $g_{\alpha\beta} = e^{2F} \eta_{\alpha\beta} \implies R^{(2)} = -2\Delta F$

$$S = \kappa \int_{\Sigma} d^2\xi \left[ \dots + 2\eta^{\alpha\beta} a_i \partial_{\alpha} x^i \partial_{\beta} F + \frac{2}{\alpha} \eta^{\alpha\beta} \partial_{\alpha} F \partial_{\beta} F \right]$$

$$\frac{1}{\alpha} = \frac{\beta^{\Phi}}{(4\pi\kappa)^2}$$

$F$  is a dynamical variable with the Liouville action as a kinetic term

- To cancel the term linear in  $F$ , we change the variables,  $F \rightarrow {}^*F = F + \frac{\alpha}{2} a_i x^i$

$$S = \kappa \int_{\Sigma} d^2\xi \left[ \left( \frac{1}{2} \eta^{\alpha\beta} {}^*G_{ij} + \epsilon^{\alpha\beta} B_{ij} \right) \partial_{\alpha} x^i \partial_{\beta} x^j + \frac{2}{\alpha} \eta^{\alpha\beta} \partial_{\alpha} {}^*F \partial_{\beta} {}^*F \right]$$

- Standard form of the action without dilaton term
- Redefined Liouville term,  $F \rightarrow {}^*F$  ( ${}^*F$  decouples)
- Redefined space-time metric,  ${}^*G_{ij} = G_{ij} - \alpha a_i a_j$
- Dilaton dependence is through the metric  ${}^*G_{ij}$
- ${}^*G_{ij}$  and the corresponding effective one  ${}^*G_{ij}^{eff}$  are singular
  - $\alpha a^2 = 1 \implies \det {}^*G_{ij} = 0$
  - $\alpha \tilde{a}^2 = 1 \implies \det {}^*G_{ij}^{eff} = 0$
 They produce gauge symmetries of the theory

- We choose Neumann boundary conditions for  ${}^*F$
- Zero central charge limit  $\alpha \rightarrow \infty \iff \beta^{\Phi} = c = 0$

## Conclusion (in presence of Liouville term)

- We carefully investigate all three cases with Liouville action

**All important results** as:

- Form of the solution
- Dimensions of Dp-brane
- Number of commutative and noncommutative variables
- Noncommutativity relation

**are the same as in the absence of Liouville action**

- **Differences** produced by Liouville term

- Local gauge symmetries appear for  $a^2 = \frac{1}{\alpha}$  and  $\tilde{a}^2 = \frac{1}{\alpha}$   
instead for  $a^2 = 0$  and  $\tilde{a}^2 = 0$
- Variables change the roles ,  $x_0 \rightarrow {}^*F$  and  $F \rightarrow x_0$ 
  - \* Instead of  $x_0$ , the variable  ${}^*F$  is commutative
  - \* In the first case  $x_0$  is noncommutative instead of  $F$
  - \* In the second and third case  $x_0$  satisfies the Dirichlet boundary conditions instead of  $F$

## Symmetries of the space-time field equations

- World-sheet metric tensor  $g_{\alpha\beta}$ , in terms of the light-cone variables  $(h^+, h^-, F)$

$$g_{\alpha\beta} = \frac{1}{2}e^{2F} \begin{pmatrix} -2h^-h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix}$$

- – World-sheet Diffeomorphisms  $\xi^\mu \rightarrow \xi^\mu + \varepsilon^\mu(\xi)$

$$\delta g^{\mu\nu} = g^{\mu\rho} \partial_\rho \varepsilon^\nu + g^{\nu\rho} \partial_\rho \varepsilon^\mu - \varepsilon^\rho \partial_\rho g^{\mu\nu}$$

- Transition to light-cone variables in terms of  $\varepsilon^\pm = \varepsilon^1 - \varepsilon^0 h^\pm$

$$\delta h^\pm = \partial_0 \varepsilon^\pm + h^\pm \partial_1 \varepsilon^\pm - \varepsilon^\pm \partial_1 h^\pm$$

$$\begin{aligned} \delta F = & -\partial_1(\varepsilon^+ + \varepsilon^-) + (\varepsilon^- - \varepsilon^+) \frac{\partial_1(h^- + h^+)}{h^- - h^+} \\ & - \frac{\varepsilon^+}{h^- - h^+} (\partial_0 F + h^- \partial_1 F) + \frac{\varepsilon^-}{h^- - h^+} (\partial_0 F + h^+ \partial_1 F) \end{aligned}$$

- Closure of algebra

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]h^\pm = \delta(\varepsilon_3)h^\pm \quad [\delta(\varepsilon_1), \delta(\varepsilon_2)]F = \delta(\varepsilon_3)F$$

- Structure functions

$$\varepsilon_3^\pm = \varepsilon_1^\pm \partial_1 \varepsilon_2^\pm - \varepsilon_2^\pm \partial_1 \varepsilon_1^\pm$$

## Classical algebra

- Generators

$$\varepsilon^\pm \rightarrow T_\pm \quad \varepsilon \odot T \equiv \int d\sigma [\varepsilon^+(\sigma)T_+(\sigma) + \varepsilon^-(\sigma)T_-(\sigma)]$$

Algebra

$$\{\varepsilon_1 \odot T, \varepsilon_2 \odot T\} = \varepsilon_3 \odot T$$

- – From particular expression for structure functions

$$\{T_\pm(\sigma), T_\pm(\bar{\sigma})\} = [T_\pm(\sigma) + T_\pm(\bar{\sigma})]\delta'$$

$$\{T_\pm, T_\mp\} = 0$$

- Note that it is Virasoro algebra with

$$\varepsilon^\pm = \varepsilon^\pm(\xi^+, \xi^-).$$

- Algebra of 2D diffeomorphisms:

two independent copies of Virasoro algebra

- Replace conformal invariance prescription by principle of 2D reparametrization invariance
- 2d metric (conformal part  $F$ ) is quantized

## Quantum algebra

- – Transition from classical to quantum theory  
Fields  $\rightarrow$  Operators

$$x^\mu \rightarrow \hat{x}^\mu, \quad \pi_\mu \rightarrow \hat{\pi}_\mu, \quad T_\pm(\varphi) \rightarrow: \hat{T}_\pm(\varphi) :$$

$$\varphi = \{G_{\mu\nu}, B_{\mu\nu}, \Phi\}$$

- Poisson brackets  $\rightarrow$  Commutators

$$\{A, B\} = C \rightarrow [\hat{A}, \hat{B}] = i\hbar\hat{C}$$

- – Breaking of Virasoro algebra (Breaking of 2D diffeomorphisms) because of normal ordering

$$[\hat{T}_\pm(\sigma), \hat{T}_\pm(\bar{\sigma})] = i\hbar[\hat{T}_\pm(\sigma) + \hat{T}_\pm(\bar{\sigma})]\delta' + (\beta_{\mu\nu}^G + \beta_{\mu\nu}^B)\hat{O}^{\mu\nu}\delta'' + \beta^\Phi\delta'''$$

$$[\hat{T}_\pm(\sigma), \hat{T}_\mp(\bar{\sigma})] = 0$$

- Quantum 2D diffeomorphisms  $\iff$   
Classical space-time equations of motions

$$\beta_{\mu\nu}^G(\varphi) = 0, \quad \beta_{\mu\nu}^B(\varphi) = 0, \quad \beta^\Phi(\varphi) = 0$$

- – It is possible to determine **exact solutions** and **exact symmetries** of the space-time field equations, which are **not known in an explicit form**
  - **Exact solutions:** String propagates in group manifold  
 $G_{\mu\nu} \rightarrow$  group metric  
 $B_{\mu\nu\rho} \rightarrow$  parallelizing torsion  
 $\Phi = const$



## Exact symmetries

- – Generalization of approach by M. Evans, B. A. Ovrut, I. Giannakis, ... based on conformal field theory
  - Replace conformal invariance prescription by principle of 2D reparametrization invariance
  - 2d metric (conformal part  $F$ ) is quantized

- – Fields variations  $\varphi \rightarrow \varphi + \delta\varphi$ 

$$\hat{T}_{\pm}(\varphi) \rightarrow \hat{T}_{\pm}(\varphi + \delta\varphi) = \hat{T}_{\pm}(\varphi) + \delta\hat{T}_{\pm}(\varphi)$$
  - If  $\varphi + \delta\varphi$  is also equation of motion

$$[\hat{T}_{\pm}(\varphi + \delta\varphi)_{\sigma}, \hat{T}_{\pm}(\varphi + \delta\varphi)_{\bar{\sigma}}] = i\hbar[\hat{T}_{\pm}(\varphi + \delta\varphi)_{\sigma} + \hat{T}_{\pm}(\varphi + \delta\varphi)_{\bar{\sigma}}]\delta'$$

$$[\hat{T}_{\pm}(\varphi + \delta\varphi)_{\sigma}, \hat{T}_{\mp}(\varphi + \delta\varphi)_{\bar{\sigma}}] = 0$$

- Conditions

$$[\hat{T}_{\pm}(\sigma), \delta\hat{T}_{\pm}(\bar{\sigma})] + [\delta\hat{T}_{\pm}(\sigma), \hat{T}_{\pm}(\bar{\sigma})] = i\hbar[\delta\hat{T}_{\pm}(\sigma) + \delta\hat{T}_{\pm}(\bar{\sigma})]\delta'$$

$$[\hat{T}_{\pm}(\sigma), \delta\hat{T}_{\mp}(\bar{\sigma})] + [\delta\hat{T}_{\pm}(\sigma), \hat{T}_{\mp}(\bar{\sigma})] = 0$$

- General solution

$$\delta\hat{T}_{\pm}(\sigma) = [\hat{\Gamma}_{\Lambda}, \hat{T}_{\pm}(\sigma)] \quad \hat{\Gamma}_{\Lambda} = \int d\sigma \Upsilon(\Lambda(x), \hat{O})$$

## Example

- – Energy-momentum tensor quadratic in currents

$$T_{\pm} = \mp \frac{1}{4\kappa} G^{\mu\nu} j_{\pm\mu} j_{\pm\nu} \quad j_{\pm\mu} = \pi_{\mu} + 2\kappa (B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}) x^{\nu'}$$

$$\delta T_{\pm} = \frac{1}{2\kappa} (\delta B_{\mu\nu} \pm \frac{1}{2} \delta G_{\mu\nu}) j_{\pm}^{\mu} j_{\pm}^{\nu}$$

- Chose the form of generator

$$\Gamma_{\Lambda} = 2\kappa \int d\sigma \Lambda_{\mu} x^{\mu'}$$

$$[\Gamma_{\Lambda}, T_{\pm}(\sigma)] = \frac{1}{2\kappa} (\partial_{\mu} \Lambda_{\nu} - \partial_{\nu} \Lambda_{\mu}) j_{\pm}^{\mu} j_{\pm}^{\nu}$$

- Symmetry transformations

$$\delta B_{\mu\nu} = \partial_{\mu} \Lambda_{\nu} - \partial_{\nu} \Lambda_{\mu} \quad \delta G_{\mu\nu} = 0$$

- We are going to find symmetries of
  - Open string theory
  - Theory in presence of dilaton field