Constructions and applications of toric CICYs

New methods in string theory and quantization (SQ2007) @ Nis, March 22-26, 2007

Maximilian Kreuzer, Vienna Univ. Tech.

Content

Toric geometry [arxiv: hep-th/0612307]

- **Definitions, homogeneous coordinates, line bundles, intersection ring, Kähler metric & symplectic quotient**
- **Calabi–Yau hypersurfaces and complete intersections**

Recent results and applications

- **Conifold transitions to non-toric Calabi–Yau varieties**
- **Torsion curves for moduli stabilization in a** `Heterotic standard model`
- **Open problems & to be done**

Ramond–Ramond background fields

- **Berkovits string and RR fields**

Collaborators

V. Batyrev (Tübingen), V. Braun (U. Penn), S. Guttenberg (Vienna)
A. Klemm (Madison), B. Ovrut (U. Penn), E. Scheidegger (Alessandria)

The Calabi–Yau tale

- SUSY compactification $\Rightarrow$ $\exists$ Killing spinor $\nabla\eta = 0$
- Kähler form $\omega_{ij} = \eta^\dagger \gamma_{[i}\gamma_{j]}\eta$, complex structure $J_i{}^k = w_{ij}g^{jk}$, $J^2 = -\mathbb{1}$
- Holomorphic 3-form $\Omega_{ijk} == \eta^\dagger \gamma_{[i}\gamma_j\gamma_{k]}\eta$, $\longrightarrow$ $b_3 = 2(h_{12}+1)$ periods

  $X_L = \int_{A_L}\Omega$ special coordinates (complex structure moduli), prepotential: $\partial_L F = F_L = \int_{B_L}\Omega$

  h_{11} Kähler moduli $\overset{\text{mirror symmetry}}{\longleftrightarrow}$ h_{12} complex structure moduli

- "large complex structure" = max. unipotent monodromy: principal period ω_0

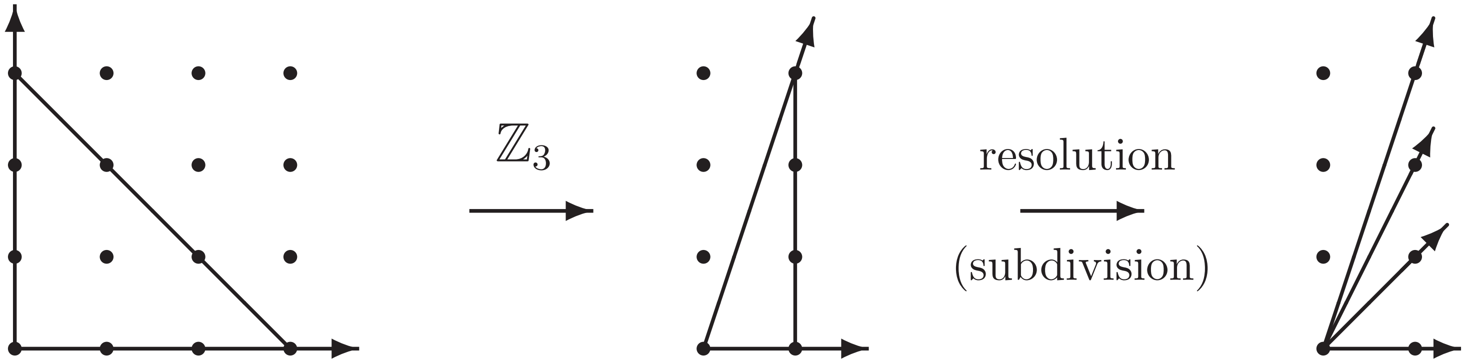
  $\omega_j^{(1)}$ log singular $\to$ $t_j = \frac{\omega_j^{(1)}(u)}{\omega_0(u)}$ defines the mirror map $q_j = e^{2\pi i t_j} = u_j + \mathcal{O}(u^2)$

  $\to$ instanton expansion (Gromov–Witten invariants)

- The quintic $\{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4$
  - 101 CS moduli $\gg$ 1 Kähler modulus
  - mirror: volum parameters $\sim$ blow-up of ambient space singularities

    $\to$ toric varieties as ambient spaces

Example: resolution of the $\mathbb{Z}_n$ singularity: $\mathbb{C}[X,Y]/\mathbb{Z}_n : \begin{smallmatrix} X\to e^{2\pi/n}X \\ Y\to e^{2\pi/n}Y \end{smallmatrix}$



Inv.: X^3, X^2Y, XY^2, Y^3
$X^6, X^5Y, X^4Y^2, \ldots$

$\tilde{X} = X^3,$
$\tilde{X}\tilde{Y} = X^2Y,$

$\tilde{X} = X_1,\ Y_1 = X_2,\ \ldots\ Y_3 = \tilde{Y}$
transition: $Y_2 = Y_1/X_1, \ldots$

Affine patch $U_\sigma \leftrightarrow$ **regular funtions** A_σ $\sim$ **semigroup** $\cong$ **cone** σ **of exponent vectors**
Global: Transition functions = Laurant monomials $\longleftrightarrow$ addition of exponent vectors

Toric variety $X = \mathbb{T} \cup D_1 \cup \ldots \cup D_r$ with **torus** $\mathbb{T}^n = (\mathbb{C}^*)^n \ni (t_1, \ldots, t_d)$
such that the $\mathbb{T}^n$ **action on itself extends to** X $\quad \mathbb{C}^* = \mathbb{C} \setminus 0 =$ complex. $S^1 = U(1)$

- **Global structure:** **fan** Σ contains faces & intersections; **affine patches** U_σ $\forall\ \sigma \in \Sigma$

  homogeneous coordinates z_i $\leftrightarrow$ **rays** $\rho_j = \langle v_j \rangle \in \Sigma^{(1)}$, **divisors** $D_j = \{z_j = 0\}$

  Rational functions $\chi_m = \prod_i t_i^{m_i} = \prod z_j^{\langle m, v_j \rangle}$ $\longleftrightarrow$ exponent vectors $m \in \mathbf{M} \cong \mathbb{Z}^d$

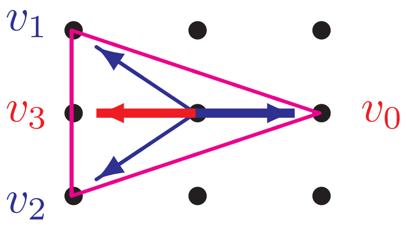
  $\sum q_j v_j = 0 \ \Rightarrow\ z_j \to \lambda^{q_j} z_j$ defines same point $(t_1, \ldots, t_d) \in \mathbb{T}^n$ $\qquad v_j \in \mathbf{N} = \mathrm{Hom}(\mathrm{M}, \mathbb{Z})$

Holomorphic quotient: $(\mathbb{C}^N - Z)/(\mathbb{C}^*)^n$ $\quad t_i = \prod z_j^{\langle e_i, v_j \rangle}$

$(z_1, \ldots, z_N) \sim (\lambda^{q_1^I} z_1, \ldots, \lambda^{q_N^I} z_N)$ for $I = 1, \ldots, n$ with $\sum q_j v_j = 0$

Example: $\mathbb{P}^{N-1} \to v_0 + \ldots + v_d = 0 \quad \longleftrightarrow \quad t_i = z_i / z_0$

GLSM → N superfields & n $U(1)$ gauge symmetries ⇒ symplectic quotient / Kähler metric

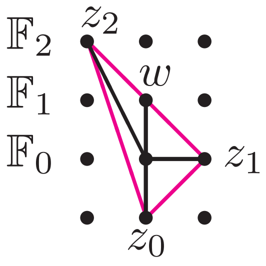


$2v_0 + v_1 + v_2 = 0 \qquad \vec{q}^{\,(1)} = (2, 1, 1, 0)$

$v_0 + v_3 = 0 \qquad \vec{q}^{\,(2)} = (1, 0, 0, 1)$

… blowup $\mathbb{F}_2$ of $\mathbb{WP}^2_{2,1,1}$ at the fix point (0:0:1) of $\lambda = -1$

exceptional set Z: $\{z_j\}$ vanish *only if* $\exists \sigma \in \Sigma : \ \{j\} \subset \sigma$



$(\lambda^2 \mu z_0, \lambda z_1, \lambda z_2, \mu w) \in (\mathbb{C}^4 - Z)/(\mathbb{C}^*)^2$

$Z = \{z_0 = w = 0\} \cup \{z_1 = z_2 = 0\} \qquad z_0 = 0 \ \Rightarrow \ 0 \neq w \to 1$

(like $\mathbb{WP}$)

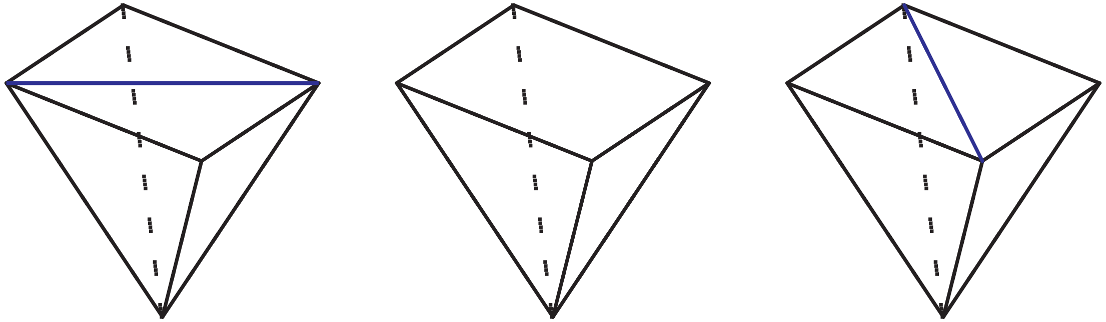
$$z_0 \neq 0 \ \Rightarrow \begin{cases} 0 \neq w \to 1 & \text{drop } z_1 = z_2 = 0 \\ w = 0 \ \Rightarrow z_0 \to 1 & \text{add } (z_1 : z_2) \end{cases}$$

The conifold singularity

The exceptional set Z depends on a fan of cones σ:

homogeneous coordinates z_i may vanish simultaneously

$\Leftrightarrow$ v_i **belong to the same cone**



small resolution: $0 \to \mathbb{P}^1 \sim S^2$

deformation $xy - uv = \varepsilon \quad 0 \to S^3$

conifold **singular geometry** **xy=uv**

$$q = (1,1,-1,-1) \qquad \to \qquad (z_0 : z_1 : z_2 : z_3) = (\lambda z_0 : \lambda z_1 : \tfrac{1}{\lambda} z_2 : \tfrac{1}{\lambda} z_3)$$

invariant = affine coordinates:

$x = \chi^{m_0} = z_0 z_2, \quad y = \chi^{m_1} = z_1 z_3, \quad u = \chi^{m_2} = z_1 z_2, \quad v = \chi^{m_3} = z_0 z_3$

Theorem: $\mathbb{P}_\Sigma$ **is regular $\Leftrightarrow$ all cones are simplicial and unimodular (=basic)**

non-simplicial: $(\mathbb{C}^*)^n$ ↔ GIT quotient (drop bad orbits) → desingularize by triangulation

simplicial fan: only abelien “orbifold” singularities → desingularize by (crepant) subdivision

Line bundles and hypersurfaces

Convex lattice polytopes Δ in M lattice:

lattice points $m \in \Delta \subset M_{\mathbb{R}}$ $\leftrightarrow$ $\{$monomials$\}$ = (Laurant) polynomial

$$f = \sum_{m \in \Delta_D \cap M} a_m \chi^m = \sum_{m \in \Delta_D \cap M} a_m \prod_j z_j^{\langle m, v_j \rangle}$$

In each patch: $f_\sigma = f \cdot \chi^{m_\sigma}$ $\rightarrow$ section of a line bundle $\mathcal{O}(D)$

Compatibilty: Cartier divisor $D = a_j D_j$ with $\langle m_\sigma, v_j \rangle \geq -a_j$ $\longleftrightarrow$ polytope Δ_D

Projectivity (Kähler): Σ is a refinement of the normal fan of Δ

Calabi–Yau hypersurfaces

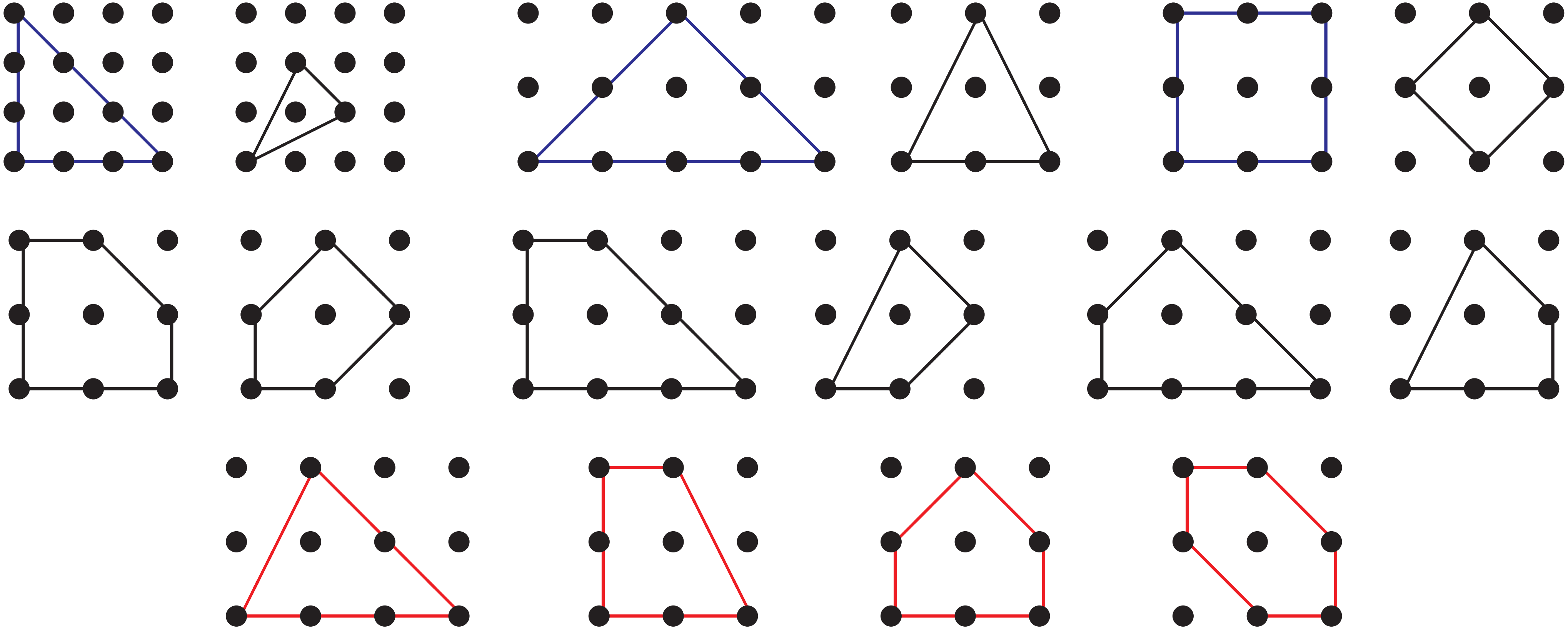
Theorem [Batyrev]: The hypersurface $\{f = 0\}$ is Calabi–Yau $(c_1 = 0)$ **if and only if the polytope Δ is reflexive, i.e. its polar polytope**

$$\Delta^\circ = \{y \in N_{\mathbb{R}} \mid \langle y, x \rangle \geq -1 \ \forall x \in \Delta \subset M_{\mathbb{R}}\}$$

is a lattice polytope (the normal fan of Δ is the fan over the faces of Δ°)

Remark: $\mathbb{P}_\Sigma$ is Fano, i.e. $c_1 > 0$, if $\mathbb{P}_\Sigma$ is smooth ($\Rightarrow$ reflexive)
$f(\frac{1}{n}\Delta)$ defines a Fano hypersurface if Δ is divisible by $n > 1$

Reflexivity



- 16 hypersurface tori (Calabi–Yau 1-folds), CY 3-folds: $\dim(\Delta) = 3 + \text{codimension} \geq 4$
- 5 Fano 2-folds (smooth, $c_1 > 0$)
- 1 Fano hypersurface: $\mathbb{P}^1$ (“hyperplane” in $\mathbb{P}^1 \times \mathbb{P}^1$)

Reflexivity & mirror symmetry

N lattice: $v_i \in \Delta^\circ$ ... homogeneous coordinates z_i,
‘toric’ (T-invariant) divisors $D_i : \{z_i = 0\}$ (e.g. $\mathbb{P}^n : D_i \sim H$)

M lattice: $m \in \Delta \to$ Monomials $\prod z_i^{\langle m,v_i\rangle+1}$
‘+1’ $\Rightarrow$ sections of a line bundle (Cartier divisor).

Batyrev ’93: generic hypersurface is CY $\Leftrightarrow$ Δ reflexive, mirror symmetry: $\Delta \longleftrightarrow \Delta^\circ$
formula for Betti numbers: count points $l(\theta)$ on dual faces $\theta \subset \Delta$ and $\theta^\circ \subset \Delta^\circ$

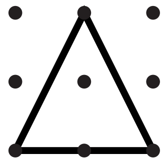
$$h_{11}(X_\Delta) = h_{2,1}(X_{\Delta^\circ}) = l(\Delta^\circ) - 1 - \dim\Delta \quad - \sum_{\operatorname{codim}(\theta^\circ)=1} l^*(\theta^\circ) + \sum_{\operatorname{codim}(\theta^\circ)=2} l^*(\theta^\circ) l^*(\theta)$$

Maximal coherent triangulation: generic CY is regular for 3 folds (singularities for 4-folds)

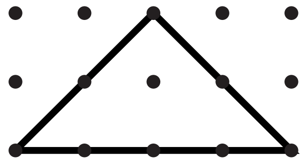
GLSM (Witten 1993): $U(1)^N$ SYM with $L = L_{kin} + L_W + L_{gauge} + L_{D,\theta}$
D-term $\Leftrightarrow$ moment map $D = -\sum q_i |z^2| - r$
$\forall U(1)$: r_j = Kähler parameters, charges q_i = ’weights’
Complex structure: coefficients in polynomials

Strominger–Yau–Zaslov: SLAG fibration with T^3 fibers → MS = T-duality
$\Delta \subset M$ = image of $\mathbb{P}_\Delta$ under moment map (symplectic reduction)
→ duality of face lattice

Nef partitions & Batyrev–Borisov duality



$\Delta^\circ \in N \quad \longrightarrow$ coordinates z_i sections $\sim \sum_{m\in\Delta}\prod_i z_i^{\langle m,v_i\rangle}$



$\Delta \in M = N^* \quad \longrightarrow$ line bundles $\leftrightarrow$ equations

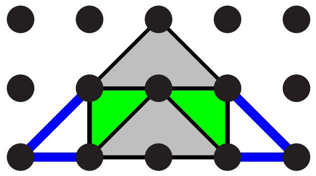
$\Rightarrow$ CICY: decompose $\Delta = \Delta_1 + \Delta_2$ (Minkowski sum)

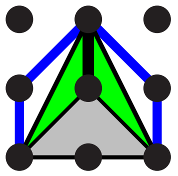
V.V. Batyrev & L.A. Borisov [alg-geom/9412017]:

- NEF partitions: piecewise linear convex "support functions" $\varphi_j(e_i) = \delta_{ij}$ numerically effective $\longrightarrow$ ample line bundles
- combinatorial duality $\leftrightarrow$ mirror symmetry ... **4 reflexive polytopes:**

$$\Delta = \Delta_1 + \Delta_2 \qquad \langle \Delta_i, \nabla_j \rangle = \begin{cases} \geq -1 & \text{if } i = j \\ \geq 0 & \text{if } i \neq j \end{cases} \qquad \Delta^* = \langle \nabla_1, \nabla_2 \rangle$$

$$\nabla^* = \langle \Delta_1, \Delta_2 \rangle \qquad \nabla = \nabla_1 + \nabla_2$$





Mirror symmetry: duality extends to Hodge data

V.V.Batyrev, L.A.Borisov: alg-geom/9509009

$$\sum(-1)^{p+q}\,h_{pq}\,t^p\,\bar{t}^q = \sum_{I=[x,y]} \frac{(-)^{\rho_x}t^{\rho_y}}{(t\bar{t})^r} S(C_x,\frac{\bar{t}}{t}) S(C_y^*,t\bar{t}) B(I;t^{-1},\bar{t})$$

- $C_x, C_y\ \in$ face lattice of Gorenstein cone spanned by (e_i,Δ_i)
- $B(I)$ encodes combinatorics of the sublattice $I=[x,y]$ with $x<y$
- $S(C_x,t)=(1-t)^{\rho_x}\sum_{m\in C_x} t^{\deg(m)}$ related to the Erhart polynomial

nef.x (∈ PALP) by Erwin Riegler [math.AG/0103214, math.CS/0204356]:

Batyrev's formula for codimension $r=1$: codim 1 - divisors do not intersect

$$h_{11}=l(\Delta^*)-1-d-\sum_{cd(\theta^*)=1} l^*(\theta^*)+\sum_{cd(\theta^*)=2} l^*(\theta^*)l^*(\theta)$$

for $r>1$ a combinatorial characterization of intersecting divisors is missing !

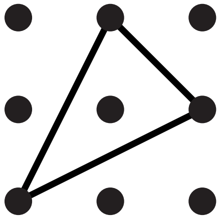
Classification results

- general Algorithm: hep-th/9505120 [M.K., H. Skarke]
- **maximal** objects are **Newton polytopes** $\sum q_i n_i = d$,
  $n_i \geq 0$, $d = \sum q_i \Rightarrow \vec{n} = \vec{1} \in \Delta$ is the only possible interior point
  $\vec{1} \in \Delta^0 \;\; \Rightarrow$ **finitely many weights** $\vec{q}$'s
- any reflexive $\Delta \;\subset\; \Delta_{max}$ comes from **combined weights**
  - simplex decomposition of 'minimal' $\Delta^* \rightarrow$ baricentric coord.
  - $1x + 1y + 1z = 3$, $1x + 1y + 2z = 4$, ${}^{1x+1y+0u+0v=2}_{0x+0y+1u+1v=2}$
  - the polytope may live on a **sublattice** (finitely many)

maximal Δ → enumerate all reflexive subpolytopes on sublattices

$\updownarrow$

minimal Δ^*

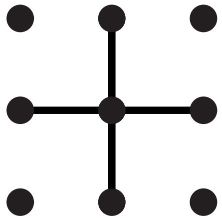


simlpex "3"

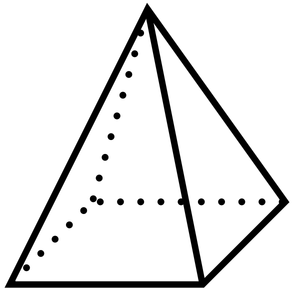
barycentric coordinates q_i

$$q_i = n_i/d$$

$$\sum n_i \vec{v}_i = 0 \ \rightarrow \qquad d = \sum n_i$$



2×2



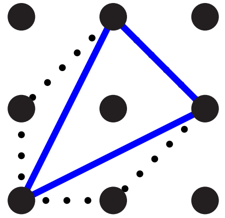
$\underbrace{3+3}$

combined weight systems (CWS)

$$\begin{array}{ccccc} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}$$

Weight vector → Newton polytope Δ_q ↔ Δ_q^*

$$\vec{q} = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) \ \rightarrow \ \Delta_q = \left\langle \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\rangle - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ \leftrightarrow$$

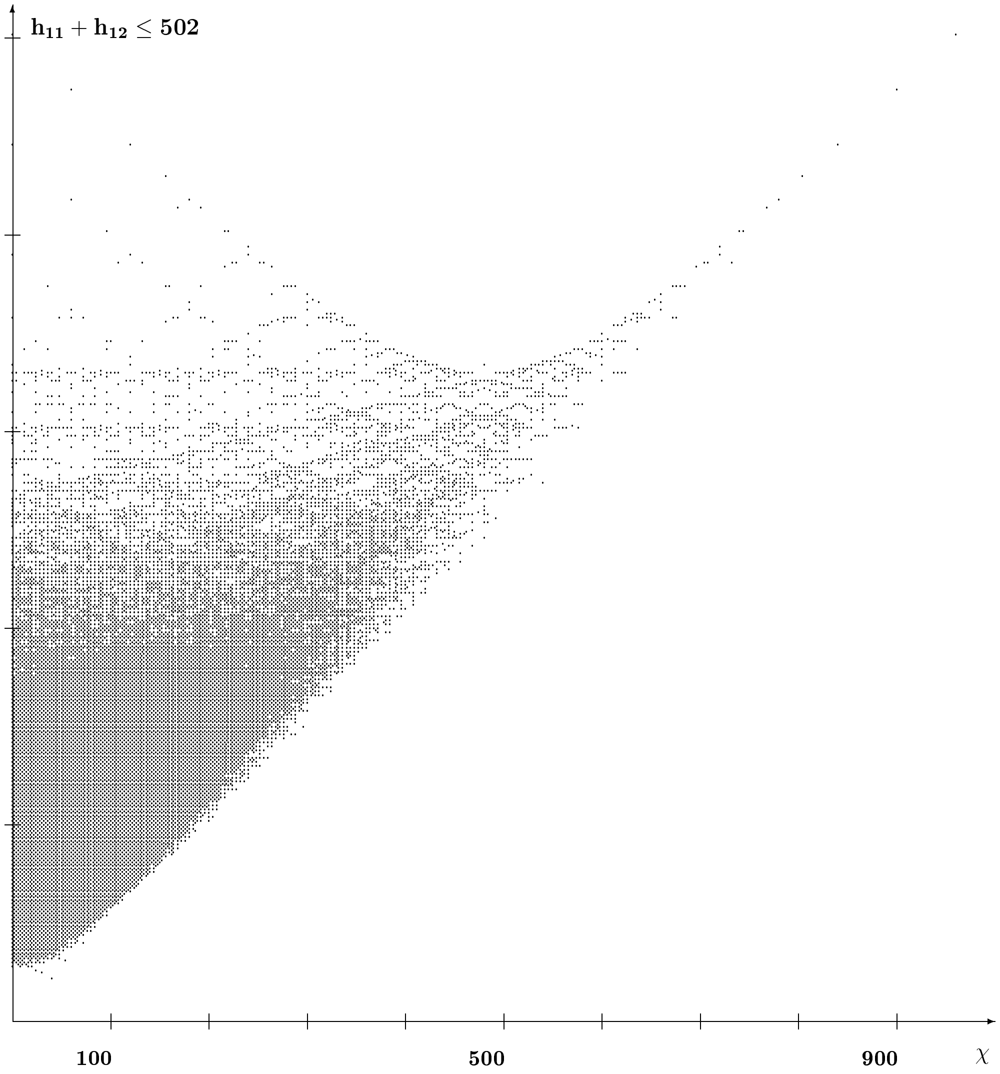


Lemma: In each dimension there is only a finite number of weights $(d, \vec{n})$ such that Δ_q has an interior lattice point.

4 dimensions: [hep-th/0002240]

- 184.026 weights, 308+25+7 maximal reflexive polyhedra
- 473.800.776 reflexive polyhedra
- 30.108 pairs of Hodge numbers
- 4.5 GB disk space → internet:

  search mask / complete database
- test: mirror symmetry !



Conifold transitions to non-toric Calabi–Yau varieties

V. Batyrev & M.K. (in preparation)

- Toric CICYs are *numerous and easy to work with!*
- Combinatorial *mirror symmetry!* [Batyrev-Borisov]: tools for computing

  quantum cohomology = Gromov-Witten = instanton sums
- But **how generic are they?**

- Reid's phantasy = Candelas: Other worlds around the corner (1990)

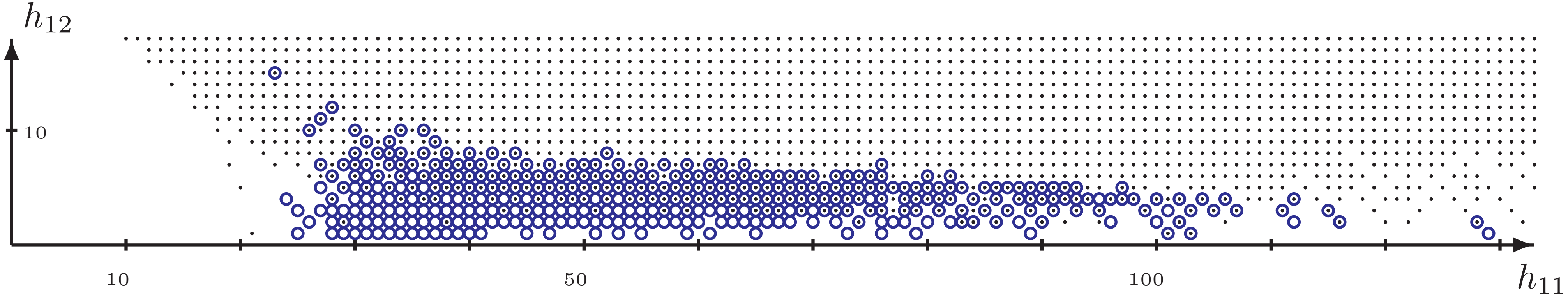
  Moduli spaces of (all?) Calabi–Yau spaces are connected by singular transitions: singular geometry, but smooth physics: Black hole condensation (Strominger 1995)

V.Batyrev, M.K. (in preparation): 4d reflexive *conifold* polytopes with ∃ smoothing deformation

$h_{11} = 1$: 8871 CYs with h12 = 21,23-51,53,55,59,61,65,73,76,79,89,101,103,129
210 smooth: h12 = 25,28-41,45,47,51,53,55,59,61,65,73,76,79,89,101,103,129

$h_{11} = 2$: 43080 CYs with h12 = 22,24-80,82-90,96,100,102,103,111,112,116,128
3470 smooth: h12 = 26,28-60,62-68,70,72,74,76,77,78,80,82-84,86,88,90,96,100,102,112,116,128

$h_{11} = 3$: . . .



Picard number $h_{11} = 1$: 210 smooth CYs with 69 different topologies

intersection numbers (topological) vs. instanton numbers (symplectic)

Picard Fuchs operators (determine periods): $\theta = t\,\frac{d}{dt}$

$$\begin{aligned}
\theta^4 &+ \frac{2}{29}\; t\; \theta(24\theta^3 - 198\theta^2 - 128\theta - 29) - \frac{4}{841}\; t^2\; (44284\theta^4 + 172954\theta^3 + 248589\theta^2 + 172057\theta + 47096) \\
&- \frac{4}{841}\; t^3\; (525708\theta^4 + 24147720\theta^3 + 44476430\theta^2 + 38390490\theta + 1275594) \\
&- \frac{8}{841}\; t^4\; (14156240\theta^4 + 79110040\theta^3 + 173954490\theta^2 + 173963590\theta + 6496262) \\
&- \frac{16}{841}\; t^5\; (\theta+1)(21520400\theta^3 + 121866360\theta^2 + 241793730\theta + 16560506) \\
&- \frac{32}{841}\; t^6\; (\theta+1)(\theta+2)(19122560\theta^2 + 91085400\theta + 11349571) \\
&- \frac{10496}{841}\; t^7\; (\theta+1)(\theta+2)(\theta+3)(56710\theta + 16301) - \frac{24529152}{841}\; t^8\; (\theta+1)(\theta+2)(\theta+3)(\theta+4)
\end{aligned}$$

Torsion in (co)homology V. Batyrev & M.K. [math.AG/0505432]

- **Mirror symmetry** exchanges h_{21} **complex structure** and h_{11} **Kähler** moduli
- What about integral cohomology?
- Universal coefficient theorem

$$\mathrm{tor}(H_i(X,\mathbb{Z})) \cong \mathrm{tor}(H^{i+1}(X,\mathbb{Z}))^*$$

- Poincaré duality:

$$\mathrm{tor}(H_i(X,\mathbb{Z})) \cong \mathrm{tor}(H^{2d-i}(X,\mathbb{Z}))$$

- 3-folds ⇒ two independent torsion groups:
  $\mathrm{tor}\, H_1(X,\mathbb{Z}) \cong \mathrm{tor}\, H^2(X,\mathbb{Z})^*$ (related to fundamental group)
  $\mathrm{tor}\, H_2(X,\mathbb{Z}) \cong \mathrm{tor}\, H^3(X,\mathbb{Z})^*$ (topological Brauer group)
- **conjecture:** exchanged under mirror symmetry
- verified for all 473 800 776 toric Calabi–Yau hypersurfaces: 16+16 cases with torsion

Torsion curves for the “Heterotic standard model”

with V. Braun, B. Ovrut and E. Scheidegger

A (3,3) paramerter example with $\pi_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$

$$\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Schoen: Fiber product of two elliptic fibers over $\mathbb{P}^1$
- $\mathbb{Z}_3$ phase (toric) × $\mathbb{Z}_3$ permutation (non-toric) free quotient
- Direct curve counting in A model limited to base-degree 1
- (permutation extension of) Batyrev–Borisov mirror
  $\Rightarrow$ B-model calculation of instantons sum
- **Surprise: Self-mirror** $\leftrightarrow$ $\mathbb{Z}_3 \times \mathbb{Z}_3$ torsion curves
- tools for computation of torsion curves (spectral sequences)
- Application: torsion curves cannot be holomorphic → SUSY breaking
  moduli stabilization (vs. Beasley–Witten): single curve in homology class!

Open problems & to be done

- **Conifold Calabi–Yau: PF operators and topology for $h_{11} > 1$**

  other topological transitions
- **5d reflexive? → F-theory, → CICYs (NEF partitions)**
  possible for limited number of points
- **direct classification of CICYs (reflexive Gorenstein cones)**
  torsion in (co)homology? conifold and other singular transitions?
- **Orientifolds and F-theory: compute intersection ring and Mori cone**
  → applications to string theory
- **SLAG submanifolds (e.g. real CYs) for M theory moduli stabilization**
  → find hyperbolic 3 manifolds

Green Schwarz String

(Informal notes prepared by S. Guttenberg)

- **Type II target-superspace $x^M = (x^m, \theta^\mu, \hat{\theta}^{\hat{\mu}})$ with global supersymmetry-transformation**

$$\begin{aligned} \delta\theta^\mu &= \varepsilon^\mu, \qquad \delta\hat{\theta}^{\hat{\mu}} = \hat{\varepsilon}^{\hat{\mu}} \\ \delta x^m &= \varepsilon\gamma^m\theta + \hat{\varepsilon}\gamma^m\theta \end{aligned}$$

SUSY-invariant one-forms (supervielbeins) in flat superspace

$$E^A \equiv \mathbf{d}x^M E_M{}^A = (\underbrace{\mathbf{d}x^a + \mathbf{d}\theta\gamma^a\theta + \mathbf{d}\hat{\theta}\gamma^a\hat{\theta}}_{\Pi^a} \quad , \quad \mathbf{d}\theta^\alpha \quad , \quad \mathbf{d}\hat{\theta}^{\hat{\alpha}})$$

- **GS-action (in conformal gauge)**

$$\begin{aligned} S_{GS} &= \int \quad \frac{1}{2}\Pi^a_z \eta_{ab} \Pi^b_{\bar{z}} + \mathcal{L}_{WZ} \\ \mathcal{L}_{WZ} &= -\frac{1}{2}\Pi^a_z \left(\theta\gamma_a\bar{\partial}\theta - \hat{\theta}\gamma_a\bar{\partial}\hat{\theta}\right) + \frac{1}{2}\left(\partial\theta\gamma^a\theta\right)\left(\hat{\theta}\gamma_a\bar{\partial}\hat{\theta}\right) - (z \leftrightarrow \bar{z}) \end{aligned}$$

- **Fermionic momenta are constrained:**

$$p_{z\alpha} = (\gamma_a\theta)_\alpha \left(\partial x^a - \frac{1}{2}\theta\gamma^a\partial\theta - \frac{1}{2}\hat{\theta}\gamma^a\partial\hat{\theta}\right) = f(\theta^\mu, \partial_1 x^m, \partial_1\theta^\mu, p_a)$$

Those fermionic constraints are called $d_{z\alpha}$

$$d_{z\alpha} \equiv p_{z\alpha} - (\gamma_a\theta)_\alpha \left(\partial x^a - \frac{1}{2}\theta\gamma^a\partial\theta - \frac{1}{2}\hat{\theta}\gamma^a\partial\hat{\theta}\right)$$

- **Constraints are mixed first (κ-symmetry)/ second class**

$$\{d_{z\alpha}(\sigma), d_{z\beta}(\sigma')\} \quad \propto \quad 2\gamma^a_{\alpha\beta}\Pi_{za}\delta(\sigma-\sigma')$$

Siegel (NPB'93): complete to a (centrally extended) closed algebra

$$\begin{aligned}
\{d_{z\alpha}, \Pi_{za}\} &\quad \propto \quad 2\gamma_{a\,\alpha\beta}\partial\theta^\beta\delta(\sigma-\sigma') \\
\{\Pi_{za}, \Pi_{zb}\} &\quad \propto \quad \eta_{ab}\delta'(\sigma-\sigma') \\
\left\{d_{z\alpha}, \partial\theta^\beta\right\} &\quad \propto \quad \delta^\beta_\alpha\delta'(\sigma-\sigma')
\end{aligned}$$

- **Same chiral algebra from the following free Lagrangian**

$$\begin{aligned}
S_{free} \quad &= \quad \int \quad \frac{1}{2}\partial x^m\eta_{mn}\bar\partial x^n + \bar\partial\theta^\alpha p_{z\alpha} + \partial\hat\theta^{\hat\alpha}\hat p_{\bar z\hat\alpha} = \\
&= \quad \int \quad \underbrace{\frac{1}{2}\Pi^a_z\eta_{ab}\Pi^b_{\bar z} + \mathcal{L}_{WZ}}_{\mathcal{L}_{GS}} + \bar\partial\theta^\alpha d_{z\alpha} + \partial\hat\theta^{\hat\alpha}\hat d_{\bar z\hat\alpha}
\end{aligned}$$

Classically coincides with GS for $d_\alpha = \hat d_{\hat\alpha} = 0$ (still mixed first-second).

Berkovits Pure Spinor String

- **Berkovits (hep-th/0001035): impement $d_\alpha = 0$ in cohomology**

$$Q = \oint \quad \lambda^\alpha d_{z\alpha}, \qquad \hat{Q} = \oint \mathbf{d}\bar{z} \quad \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\bar{z}\hat{\alpha}}$$

d_α **is not pure first class** $\Leftrightarrow$ $Q^2 = 0$ **requires pure spinor constraint** $\lambda\gamma^a\lambda = 0$

- **Berkovits pure spinor string action in flat background (add only $\mathcal{L}_{gh}$)**

$$\begin{aligned}
S_{ps} &= \int \quad \underbrace{\frac{1}{2}\Pi^a_z \eta_{ab} \Pi^b_{\bar{z}} + \mathcal{L}_{WZ}}_{\mathcal{L}_{GS}} + \bar{\partial}\theta^\alpha d_{z\alpha} + \partial\hat{\theta}^{\hat{\alpha}} \hat{d}_{\bar{z}\hat{\alpha}} + \mathcal{L}_{gh} \\
\mathcal{L}_{WZ} &= -\frac{1}{2}\Pi^a_z \left(\theta\gamma_a\bar{\partial}\theta - \hat{\theta}\gamma_a\bar{\partial}\hat{\theta}\right) + \frac{1}{2}\left(\partial\theta\gamma^a\theta\right)\left(\hat{\theta}\gamma_a\bar{\partial}\hat{\theta}\right) - (z \leftrightarrow \bar{z}) \\
\Pi^a_z &= \partial x^a + \partial\theta\gamma^a\theta + \partial\hat{\theta}\gamma^a\hat{\theta} \\
\mathcal{L}_{gh} &= \bar{\partial}\lambda^\alpha \omega_{z\alpha} + \partial\hat{\lambda}^{\hat{\alpha}} \omega_{\hat{\alpha}} + L_{z\bar{z}a}(\lambda\gamma^a\lambda) + \hat{L}_{z\bar{z}a}(\hat{\lambda}\gamma^a\hat{\lambda})
\end{aligned}$$

- Lagrange multiplier L good enought at classical level. Quantization of (λ, ω) is more tricky
- Pure spinor constraint (first class) generates antighost gauge symmetry

$$\delta_{(\mu)}\omega_{z\alpha} = \mu_{za}(\gamma^a\lambda)_\alpha$$

Type II PS String in General Background

Berkovits&Howe [hep-th/0112160]: deform by vertex; Bedoya&Chandía [hep-th/0609161]: 1-loop; Guttenberg'06: type II BRST-transformations

- **Curved background: up to field redefinitions most general renormalizable action with ghostnumber 0 is (with $G_{MN} = E_M{}^a e^{2\Phi} \eta_{ab} E_N{}^b$):**

$$\begin{aligned} S \;=\; \int \quad & \frac{1}{2}\partial x^M \left(G_{MN}(x) + B_{MN}(x)\right)\bar\partial x^N + \bar\partial x^M E_M{}^\alpha(x)\, d_{z\alpha} + \partial x^M E_M{}^{\hat\alpha}(x)\, \hat d_{\bar z\hat\alpha} + \mathcal{T}(x) \\ & + d_{z\alpha}\mathcal{P}^{\alpha\hat\beta}(x)\, \hat d_{\bar z\hat\beta} + \mathcal{E}^\alpha C_\alpha{}^{\beta\hat\gamma}(x)\, \omega_{z\beta}\hat d_{\bar z\hat\gamma} + \hat{\mathcal{E}}^{\hat\alpha}\hat C_{\hat\alpha}{}^{\hat\beta\gamma}(x)\, \hat\omega_{\bar z\hat\beta} d_{z\gamma} + \mathcal{E}^\alpha\hat{\mathcal{E}}^{\hat\alpha} S_{\alpha\hat\alpha}{}^{\beta\hat\beta}(x)\, \omega_{z\beta}\hat\omega_{\bar z\hat\beta} \\ & + \underbrace{\left(\bar\partial\mathcal{E}^\beta + \mathcal{E}^\alpha\bar\partial x^M \Omega_{M\alpha}{}^\beta(x)\right)}_{\equiv\nabla_{\bar z}\mathcal{E}^\beta}\omega_{z\beta} + \underbrace{\left(\partial\hat{\mathcal{E}}^{\hat\beta} + \hat{\mathcal{E}}^{\hat\alpha}\partial x^M \hat\Omega_{M\hat\alpha}{}^{\hat\beta}(x)\right)}_{\equiv\hat\nabla_z\mathcal{E}^{\hat\beta}}\hat\omega_{\bar z\hat\beta} \; + L_{z\bar z a}(\mathcal{E}\gamma^a\mathcal{E}) + \\ & \qquad + \hat L_{z\bar z\hat a}(\hat{\mathcal{E}}\gamma^{\hat a}\hat{\mathcal{E}}) \end{aligned}$$

- **A nonconstant tachyon background $\mathcal{T}$ will not be BRST-invariant (flat:** $s\mathcal{T} = \mathcal{E}^\alpha\nabla_\alpha\mathcal{T} \overset{!}{=} 0 \quad \Rightarrow [\nabla_\alpha, \nabla_\beta]\mathcal{T} = -2\gamma^a_{\alpha\beta}\nabla_a\mathcal{T} \overset{!}{=} 0 \quad \Rightarrow \mathcal{T} \overset{!}{=} const$**)**

- **The general ansatz for the BRST-currents (gh#1, conf weight 1) in the curved background can (reparametrizing d) be brought to**

$$\begin{aligned} \mathbf{j}_z &= \mathcal{E}^\alpha d_{z\alpha} + \mathcal{E}^\alpha W_{\alpha M}(x)\,\partial_z x^M, \quad \mathbf{j}_{\bar z} = 0, \qquad\qquad \left(Q = \oint \mathbf{d}z \quad \mathbf{j}_z\right) \\ \hat\jmath_{\bar z} &= \hat{\mathcal{E}}^{\hat\alpha}\hat d_{\bar z\hat\alpha} + \hat{\mathcal{E}}^{\hat\alpha}\hat W_{\hat\alpha M}(x)\,\partial_{\bar z} x^M, \quad \hat\jmath_z = 0 \end{aligned}$$

Reparametrizations

- **Antighost gauge symmetry restricts the form of Ω:**

$$\Omega_{M\alpha}{}^{\beta} = \frac{1}{2}\Omega_M \delta_\alpha{}^{\beta} + \frac{1}{4}\Omega_{M a_1 a_2}\gamma^{a_1 a_2}{}_{\alpha}{}^{\beta}$$

  ⇒looks like connection for local Lorentz and local scale transformations

- concept of reparametrization invariances:
  - reparametrize ws-fields & background fields
  - no ws-symmetry (transformation of “coupling constants”)!
  - but will lead to same constraints for transformed background fields ⇒ target space sym.

- some reps already used to eliminate background fields. Remaining reps
  - x^M: correspond to targetspace superdiffeomorphisms
  - $\mathcal{E}^\alpha$: restricted to leave $L_{z\bar{z}a}(\mathcal{E}\gamma^a\mathcal{E})$ invariant (no compensating background field)
    - ⇒local Lorentz transformations & local scale transformations
    - coupled to rep of $L_{z\bar{z}a}$, $\omega_{z\alpha}$ (kinetic ghost term) and $d_{z\alpha}$ (BRST-operator)
    - $\Omega_{M\alpha}{}^{\beta}$ plays the role of a structure group connection
  - $\hat{\mathcal{E}}^{\hat{\alpha}}$: **independent** local Lorentz transformation
  - reparametrization of (auxiliary) $E_M{}^a$: yet another indep local Lorentz & scale trafo

- Lorentz transformations will be coupled by gauge fixing some torsion components

BRST invariance

- $\mathbf{j}_z$ **and** $\hat{\jmath}_{\bar{z}}$ **have to correspond to nilpotent symmetry trafos:**
  - **symmetry**$\Leftrightarrow$**on-shell conserved current:** $\bar{\partial}\mathbf{j}_z = -\mathbf{s}\varphi^I \frac{\delta}{\delta\varphi^I} S$ **(Lagrangian approach!)**
  - **on-shell nilpotency** $\Leftrightarrow \mathbf{s}^2\varphi^I = A^{IJ}\frac{\delta}{\delta\varphi^J}S +$ **antigh-gauge**$\Leftrightarrow \mathbf{s}\mathbf{j}_z \propto (\lambda\gamma^a\lambda)$
- **Instead one gets**

$$\begin{aligned} \bar{\partial}\mathbf{j}_z &= -\mathbf{s}\varphi^I \frac{\delta}{\delta\varphi^I} S + \ldots \\ \mathbf{s}\mathbf{j}_z &\propto (\lambda\gamma^a\lambda) + \ldots \end{aligned}$$

  - **First equ: Determine at the same time constraints on the background fields** $(\ldots \overset{!}{=} 0)$ **and the BRST-transformations of all the worldsheet fields** φ^I
  - **Lengthy calculation! Introduce spacetime covariant variation:** $\delta_{cov}\mathcal{E}^\alpha \equiv \delta\mathcal{E}^\alpha + \delta x^M \Omega_{M\beta}{}^\alpha \mathcal{E}^\beta$ **etc.**
- **Additional terms** $(\ldots)$ **of both equ's have to vanish for consitency** $\Rightarrow$ **constraints on background fields:**
  - $W_{\alpha M} = 0 \quad \Rightarrow \quad \mathbf{j}_z = \mathcal{E}^\alpha d_{z\alpha}$ **(same as in flat background)**
  - $C_\alpha{}^{\beta\hat{\gamma}} = \nabla_\alpha \mathcal{P}^{\beta\hat{\gamma}}, \qquad S_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = -\nabla_\alpha \hat{C}_{\hat{\alpha}}{}^{\hat{\beta}\beta} + 2\hat{R}_{\alpha\hat{\gamma}\hat{\alpha}}{}^{\hat{\beta}}\mathcal{P}^{\beta\hat{\gamma}}$ **(**$\hat{R}$ **defined via** $\hat{\Omega}$**)**
  - **SUGRA constraints on** $H \equiv \mathbf{d}B$, $T^A \equiv \mathbf{d}E^A - E^C \wedge \Omega_C{}^A$ **and** $R_A{}^B \equiv \mathbf{d}\Omega_A{}^B - \Omega_A{}^C \wedge \Omega_C{}^B \qquad \longrightarrow$

SUGRA constraints

- **Resulting constraints can be simplified.**
  - **use remaining residual rep invariance of $d_{z\alpha}$ (local shift symmetry) to fix $T_{\alpha\beta}{}^{\gamma} = \hat{T}_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} = 0$**
  - **use 2 of 3 local scale & Lorentz-invariances to fix $T_{\alpha\beta}{}^{c} = \gamma^{c}_{\alpha\beta}$ and $T_{\hat{\alpha}\hat{\beta}}{}^{c} = \gamma^{c}_{\hat{\alpha}\hat{\beta}}$**
  - **have to check Bianchi identities for H and T afterwards (lengthy!)**
- **$\Omega_a = \nabla_a \Phi, \quad \Omega_{\hat{\alpha}} = \nabla_{\hat{\alpha}} \Phi$ plus $\Omega_\alpha = \nabla_\alpha \Phi_{Dil}$ (quantum BRST-invariance, Berk/Howe '01)**
  Fix remaining local scale invariance: $\Phi = \Phi_{Dil} \quad \Rightarrow \nabla_A \Phi = \Omega_A$
- **Nonvanishing components of H are H_{abc} and $H_{c\alpha\beta} = \gamma_{c\,\alpha\beta}$ and the nonvanishing components of T are $T_{ab}{}^{\gamma} = \frac{1}{16} \nabla_{\hat{\gamma}} \mathcal{P}^{\gamma\hat{\delta}} \cdot e^{2\Phi} \gamma_{ab\,\hat{\delta}}{}^{\hat{\gamma}}$,$T_{\hat{\alpha}b}{}^{\gamma} = e^{2\Phi} \gamma_{b\hat{\alpha}\hat{\gamma}} \mathcal{P}^{\gamma\hat{\gamma}}$ and**

$$T_{ab}{}^{c} \quad = \quad \frac{1}{2} H_{ab}{}^{c}, \qquad \hat{T}_{ab}{}^{c} = -\frac{1}{2} H_{ab}{}^{c}$$

- **Several algebraic constraints for the curvature like $R_{\hat{\alpha}\hat{\beta}\alpha}{}^{\beta} = R_{[\alpha\beta\gamma]}{}^{\delta} = 0$ but also numerous differential equations like $\nabla_{[a} T_{bc]}{}^{\delta} = -3 H_{[ab|}{}^{e} T_{e|c]}{}^{\delta} - 2 \hat{T}_{[ab|}{}^{\hat{\varepsilon}} e^{2\Phi} \gamma_{|c]\,\hat{\varepsilon}\hat{\delta}} \mathcal{P}^{\delta\hat{\delta}}$ or**

$$\nabla_\alpha \mathcal{P}^{\alpha\hat{\delta}} = 0$$

BRST-transformations

The resulting covariant BRST transformations (e.g. $s_{cov}\mathcal{E}^{\alpha} \equiv s\mathcal{E}^{\alpha} + sx^{M}\Omega_{M\beta}{}^{\alpha}\mathcal{E}^{\alpha}$) look as follows

$$\begin{aligned}
sx^{M} &= \mathcal{E}^{\alpha}E_{\alpha}{}^{M}, \qquad \hat{s}x^{M} = \hat{\mathcal{E}}^{\hat{\alpha}}E_{\hat{\alpha}}{}^{M} \\
s_{cov}\mathcal{E}^{\alpha} &= 0, \qquad \hat{s}_{c\hat{o}v}\hat{\mathcal{E}}^{\hat{\alpha}} = 0 \\
s_{cov}\omega_{z\alpha} &= P_{z\alpha}, \qquad \hat{s}_{c\hat{o}v}\hat{\omega}_{\bar{z}\hat{\alpha}} = \hat{P}_{\bar{z}\hat{\alpha}} \\
s_{cov}P_{z\delta} &= -2\mathcal{E}^{\alpha}\Pi_{z}^{c}\tilde{\gamma}_{c\,\alpha\delta} + \mathcal{E}^{\alpha}\mathcal{E}^{\alpha_2}R_{\alpha_2\alpha\delta}{}^{\beta}\omega_{z\beta} \\
\hat{s}_{c\hat{o}v}P_{\bar{z}\hat{\delta}} &= -2\hat{\mathcal{E}}^{\hat{\alpha}}\Pi_{\bar{z}}^{c}\tilde{\gamma}_{c\,\hat{\alpha}\hat{\delta}} + \hat{\mathcal{E}}^{\hat{\alpha}}\hat{\mathcal{E}}^{\hat{\alpha}_2}\hat{R}_{\hat{\alpha}_2\hat{\alpha}\hat{\delta}}{}^{\hat{\beta}}\omega_{\bar{z}\hat{\beta}} \\
s_{c\hat{o}v}\hat{P}_{\bar{z}\hat{\gamma}} &= -2\mathcal{E}^{\alpha}\hat{\mathcal{E}}^{\hat{\alpha}}\hat{R}_{\alpha\hat{\gamma}\hat{\alpha}}{}^{\hat{\beta}}\hat{\omega}_{\bar{z}\hat{\beta}} \\
\hat{s}_{cov}P_{z\gamma} &= -2\hat{\mathcal{E}}^{\hat{\alpha}}\mathcal{E}^{\alpha}R_{\hat{\alpha}\gamma\alpha}{}^{\beta}\omega_{z\beta} \\
s_{cov}L_{z\bar{z}a} &= \frac{1}{8}\gamma_{a}^{\alpha_3\alpha_4}X_{\alpha_3\alpha_4}, \qquad \hat{s}_{c\hat{o}v}\hat{L}_{z\bar{z}a} = \frac{1}{8}\gamma_{a}^{\hat{\alpha}_3\hat{\alpha}_4}X_{\hat{\alpha}_3\hat{\alpha}_4}
\end{aligned}$$

$X_{\alpha\beta} = \ldots$ **(lengthy)**

SUSY-transformation

- Superdiffeos $\xi^A = \big(\xi^a(x,\theta), \xi^\alpha(x,\theta), \xi^{\hat\alpha}(x,\theta)\big)$ contain local SUSY and spacetime diffeos at the $\theta = 0$ components of the parameters.
- Fix part of auxiliary gauge freedom by going to **Wess-Zumino like gauge**

$$E_M{}^A\big|_{\theta=0} = \begin{pmatrix} e_m{}^a & \psi_m{}^\alpha & \hat\psi_m{}^{\hat\alpha} \\ 0 & \delta_\mu{}^\alpha & 0 \\ 0 & 0 & \delta_{\hat\mu}{}^{\hat\alpha} \end{pmatrix}$$

$$\Omega_{mA}{}^B\big| = \omega_{mA}{}^B, \qquad \Omega_{\mathcal{M}A}{}^B\big| = \begin{pmatrix} \Omega_{\mathcal{M}}\delta_a^b & 0 & 0 \\ 0 & \frac{1}{2}\Omega_{\mathcal{M}}\delta_\alpha{}^\beta & 0 \\ 0 & 0 & \frac{1}{2}\hat\Omega_{\mathcal{M}}\delta_{\hat\alpha}{}^{\hat\beta} \end{pmatrix}$$

- Define SUSY transformations as the fermionic stabilizer of this gauge, which leads for example to

$$\begin{aligned} \delta\psi_m{}^\alpha &= \partial_m\xi_0^\alpha + \omega_{m\gamma}{}^\alpha\xi_0^\gamma + 2\xi_0^{\hat\gamma}\,T_{\hat\gamma m}{}^\alpha\big| = \\ &= \nabla_m\xi_0^\alpha + 2e^{2\Phi}\xi_0^{\hat\gamma}\gamma_{a\,\hat\gamma\hat\delta}P^{\alpha\hat\delta}e_m{}^a \\ \delta\lambda_\alpha &= -2\xi_0^\gamma\gamma^a_{\gamma\alpha}\nabla_a\Phi + \frac{1}{2}\xi^{\mathcal{C}}\lambda_{\mathcal{C}}\lambda_{\mathcal{A}} + \xi_0^\gamma\nabla_\alpha\nabla_\gamma\Phi\big| \end{aligned}$$

  where $\lambda_\alpha = \Omega_\alpha| = \nabla_\alpha\Phi|$. Connection contains H-field (not just Levi-Civita)

- $\delta\psi = \delta\lambda = 0$ ⇒ generalized CY in compactification manifold
  [Grana, Minasian, Petrini, Tomasiello, hep-th/0505212]