

Non-topological non-commutativity in string theory

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with M. Herbst, A. Kling: hep-th/0106156, hep-th/0202077, hep-th/0312043

+ work in progress: with S. Guttenberg, M. Herbst, R. Rashkov

& Q: help from experts on NC geometry in the audience ;-)

Contents:

- Deformation quantization: From Moyal to Kontsevich
- Open strings, BI electrodynamics and non-commutativity
- Beyond the topological limit: derivative expansion
- Associativity and effective actions
- Superstrings and non-anticommutative superspace

Deformation quantization

- Phase space $\mathcal{M}(p_i, q^j)$: Poisson bracket $\{p_i, q^j\}_{PB} = \delta_i^j$
 - general coordinates $M(x^\mu)$: $\{f, g\}_{PB} = \Theta^{\mu\nu}(x) \partial_\mu f \partial_\nu g$
 - Jacobi id. $\sum \{\{f, g\}_{PB}, h\}_{PB} = 0 \Leftrightarrow \sum \theta^{\mu\alpha} \partial_\alpha \theta^{\nu\rho} = 0$
 - Polyvector fields $\Lambda^\bullet TM$: vector fields $\xi \in \Lambda^1 T$, **bi-vector** $\Theta \in \Lambda^2 TM, \dots$
 - Schouten–Nijenhuis bracket: extends Lie bracket to **graded bi-derivation** on polyvector fields, i.e. $[T, \cdot]$ and $[\cdot, T]$ are **left/right (anti)derivations**
- $$[\xi, T] = \mathcal{L}_\xi T, \quad [\theta, \theta]^{\mu\nu\rho} = \frac{2}{3} \sum \theta^{\mu\alpha} \partial_\alpha \theta^{\nu\rho} \in \Lambda^3 TM$$
- **Deformation Quantization** \sim Associative “operator product”
 - $f \cdot g \rightarrow (f \star g)(x) = fg + \frac{i}{2} \hbar \{f, g\}_{PB} + \mathcal{O}(\hbar^2) \in C^\infty[[\hbar]]$
- **Moyal product**: $= \exp\left(\frac{i}{2} \hbar \Theta^{\mu\nu} \partial_{y^\mu} \partial_{z^\nu}\right) f(y)g(z) \Big|_{y=z=x}$
 - for constant Θ ! ... what about change of coordinates?

- Symplectic case: $\det \Theta \neq 0 \Rightarrow [\Theta, \Theta] = 0 \Leftrightarrow d\omega = 0$ with $\omega = \Theta^{-1}$
existence: De Wilde + Lecompte '83; **constructive:** Fedosov '85
- Poisson case: Kontsevich [q-alg/9709040]
 - **Formality map** U : Polyvector $T \rightarrow$ Poly-DifferentialOperator D
 - $U(T_1, \dots, T_n) = \sum_{\Gamma} w_{\Gamma} D_{\Gamma}$ $w_{\Gamma} =$ convergent integrals (string-inspired)
 - Cattaneo–Felder: = topological σ -model perturbative expansion
 - **Quasi-isomorphism of \mathcal{L}_{∞} algebras**
 Hochschild complex (cf. Ivo's)
 Schouten–Nijenhuis \mapsto Gerstenhaber bracket (=“commutator” of PDOs)

Associativity, gauge equivalence & Hochschild complex

- Let $f \star g = fg + \hbar B_1(f, g) + \mathcal{O}(\hbar^2)$ with $B_1(f, g) = B^{\mu\nu} f_\mu g_\nu$, $f_\mu = \partial_{x^\mu} f$
- “associator” \sim Hochschild co-cycle:

$$f \star (g \star h) - (f \star g) \star h = \hbar (f B_1(g, h) - B_1(fg, h) + B_1(f, gh) - B_1(f, g)h) + \hbar^2 \dots$$

- gauge equivalence: $f \rightarrow Df$ with $D(1) = 1 \Rightarrow$

$$D = 1 + \hbar (D_1^\mu \partial_\mu + D_1^{\mu\nu} \partial_\mu \partial_\nu + \dots) + \hbar^2 \dots,$$

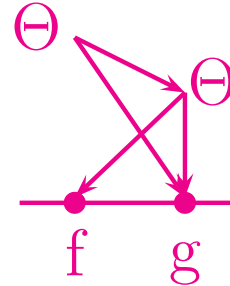
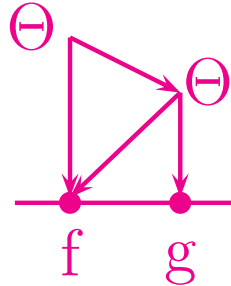
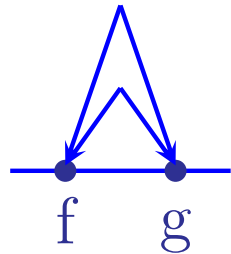
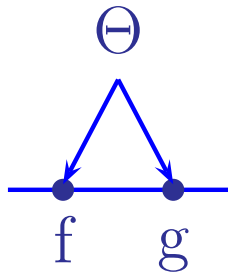
$$f \star' g = D(D^{-1} f \star D^{-1} g) = f \star g \Rightarrow$$

$$B'_1(f, g) - B_1(f, g) = -f D_1(g) + D_1(fg) - D_1(f)g \dots \text{coboundary}$$

- $D_1 = D^{\mu\nu} \partial_\mu \partial_\nu$ removes symmetric $B^{(\mu\nu)}$ \rightarrow choose $B^{\mu\nu} = \Theta^{\mu\nu} \in \Lambda^2$

define $[f \star g] \equiv fg + i\frac{\hbar}{2}\Theta^{\mu\nu}f_\mu g_\nu - \frac{\hbar^2}{8}\Theta^{\mu\alpha}\Theta^{\nu\beta}f_{\mu\alpha}g_{\nu\beta} \dots$ “Moyal part”

$$\Rightarrow f \star g = [f \star g] - \hbar^2 \Theta^{\mu\rho} \partial_\rho \Theta^{\alpha\beta} (a [f_{\alpha\mu} \star g_\beta] + b [f_\beta \star g_{\alpha\mu}]) + \mathcal{O}((\partial\Theta)^2)$$



– f, g boundary insertions (open strings)

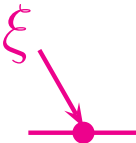
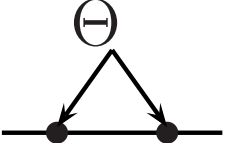
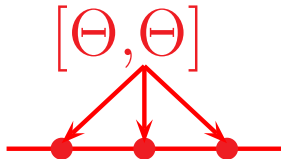
– integration over Θ 's in upper half plane (vertices)

• $f \star (g \star h) - (f \star g) \star h = 0$ at order \hbar^2 (only need to keep $\Theta \partial \Theta$ terms!)

– if and only if $a = b = \frac{1}{12}$ and $[\Theta, \Theta] = 0$

\Rightarrow associativity requires Poisson structure (mod gauge equivalence)

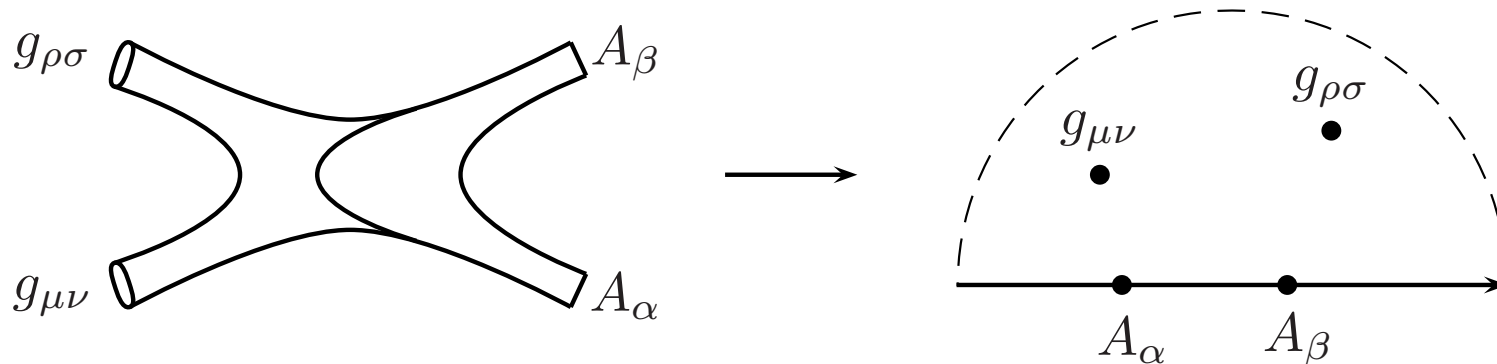
Formality, associativity and diffeomorphisms

- Formality:   
 - $[\Theta, \Theta] \mapsto$ Gerstenhaber bracket $[\star, \star]$ of $\star(f, g) \equiv$ associator \Rightarrow this proves associativity for Poisson structures Θ
 - $\Theta^{\mu\nu} f_\mu g_\nu \mapsto f \star g \dots$ this defines the Kontsevich product
 - $\xi \rightarrow D_\xi = \mathcal{L}_\xi + \mathcal{O}(\hbar)$ deformed Lie derivative !
- Formality $\Rightarrow \partial_t(f \star_t g) = D_\xi f \star g + f \star D_\xi g - D_\xi(f \star g)$ with $\dot{x}^\mu = \xi^m$
- coordinate transformation entails gauge transformation !
 - used by Cattaneo–Felder to “globalize” the \star product

Open strings, gauge invariance & electrodynamics

Closed string (conformal gauge): $\mathcal{L} = \frac{1}{2\pi\alpha'} \partial X^\mu \bar{\partial} X^\nu \left(g_{\mu\nu}(X) + B_{\mu\nu}(X) \right)$

String diagram for **photon–graviton scattering**:



- Gauge invariance: $\delta_\Lambda \left(\int_\Sigma X^* B \right) = 0$ for $\delta_\Lambda B = d\Lambda$ (Stokes' theorem)

But: open strings \rightarrow boundary term $\delta_\Lambda \left(\int_\Sigma B \right) = \int_{\partial\Sigma} \Lambda$

- Introduce **compensator field** (\sim surface charge in E-dyn.):

$$S_A = \int_{\partial\Sigma} A = \int_{\partial\Sigma} ds \partial_s X^\mu A_\mu(X) = \int_\Sigma F \quad \text{with} \quad F = dA$$

Gauge invariance: $\delta B = d\Lambda, \quad \delta A = -\frac{1}{2\pi\alpha'} \Lambda + d\lambda \Rightarrow$

Invariant field strengths: $H = dB = d\mathcal{F}, \quad \mathcal{F} = B + 2\pi\alpha' F.$

- D-brane effective action ... $\mathcal{L}_{BI} \sim \sqrt{|g_{\mu\nu} + \mathcal{F}_{\mu\nu}|}$... gauge invariant
- Superstrings: $A \rightarrow$ superpartner gaugino λ

Boundary conditions and noncommutativity

- strings \rightarrow operators $X^\mu(\sigma, \tau) \rightarrow$ NC operator product

$$S = \frac{1}{2\pi\alpha'} \int_{\mathbb{H}} d^2z \partial X^\mu \bar{\partial} X^\nu \left(g_{\mu\nu}(X) + \mathcal{F}_{\mu\nu}(X) \right), \quad \mathcal{F} = B + 2\pi\alpha' dA$$

- **Surface terms** ... fluctuations about background value $X \rightarrow X + \xi$:

$$\begin{aligned} & \partial(\delta X^\mu \bar{\partial} X^\nu (g_{\mu\nu} + \mathcal{F}_{\mu\nu})) + \bar{\partial}(\partial X^\mu \delta X^\nu (g_{\mu\nu} + \mathcal{F}_{\mu\nu})) \\ &= \partial(\delta X^\mu \bar{\partial} X^\nu (g_{\mu\nu} + \mathcal{F}_{\mu\nu})) + \bar{\partial}(\partial X^\nu \delta X^\mu (g_{\mu\nu} - \mathcal{F}_{\mu\nu})) \end{aligned}$$

- **Boundary conditions:** $\int dy \partial(\cdot) = - \int dy \bar{\partial}(\cdot)$

$$g_{\mu\nu}(\partial - \bar{\partial})X^\nu - \mathcal{F}_{\mu\nu}(\partial + \bar{\partial})X^\nu = g_{\mu\nu} \partial_{\text{tang.}} X^\nu - \mathcal{F}_{\mu\nu} \partial_{\text{normal}} X^\nu = 0$$

Propagator:

$$\langle \zeta^\mu(u, \bar{u}) \zeta^\nu(w, \bar{w}) \rangle_{\mathcal{F}} = -\alpha' \left\{ g^{\mu\nu} (\ln |u - w| - \ln |u - \bar{w}|) + G^{\mu\nu} \ln |u - \bar{w}|^2 - \Theta^{\mu\nu} \ln \left(\frac{\bar{w} - u}{\bar{u} - w} \right) \right\}$$

- define $(g_{\mu\nu} + \mathcal{F}_{\mu\nu})^{-1} = G^{(\mu\nu)} + \Theta^{[\mu\nu]}$
- $g^{\mu\nu}$ -term $\rightarrow 0$ at boundary $\partial\Sigma \Rightarrow G^{\mu\nu} =$ “open string metric”
- for $G^{\mu\nu} \rightarrow 0$ propagator \rightarrow antisym. step function at $\partial\Sigma$
 \Rightarrow non-commutative non-singular operator product \sim Moyal
- **Proposal:** define non-commutative product

$$f(x) \circ g(x) = \frac{1}{\sqrt{|g+\mathcal{F}|}} \int \mathcal{D}\xi e^{-S[X=x+\xi]} f(X(0)) g(X(1))$$

- includes quantum corrections; regularized by putting $u = 0, w = 1$
- off-shell A_∞ administrating non-associativity
- BI measure comes up in sum over graphs

Philosophy:

- Non-commutative product \equiv **summing up leading** strong B/F field effects
- Beyond topological limit (Seiberg-Witten; Kontsevich) **non-associative**
- Cyclicity = **Connes-Flato-Sternheimer** = (strongly) closed product w.r.t. **measure Ω**

$$\int \Omega f \circ g = \int \Omega f \cdot g \quad \text{and} \quad \int \Omega (f \circ g) \circ h = \int \Omega f \circ (g \circ h)$$

- **Variation of effective action:** cyclicity \Rightarrow associativity not necessary

$$\delta \int \Omega (\phi \circ ((\phi \circ \phi) \circ \phi)) = \int \Omega \delta \phi \cdot (\dots)$$

and: associativity not sufficient; cyclicity (partly) fixes gauge equivalence

- Felder Shoikhet '00: **Kontsevich product cyclic if $\partial_\mu(\Omega\Theta^{\mu\nu}) = 0$**
– but not canonical: \nexists Liouville measure for Poisson structure
- Maxwell equation: D – branes suggest to treat **bulk fields as background** and to **impose generalized Maxwell equation:**

$$\partial_\mu(\sqrt{\det(g + \mathcal{F})}\Theta^{\mu\nu}) = 0 \quad \Leftrightarrow \quad G^{\rho\sigma} D_\rho \mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma} H_{\rho\sigma}{}^\lambda \mathcal{F}_{\lambda\mu} = 0$$

How does it work?

- **Derivative expansion:** Exact to **all order in θ** , expand in $\partial\Theta$
- **Tachyon** effective action (cf. **Boundary SFT**, Ivo's talk)
 - **2-point function** $\rightarrow f \circ g$; structure of **3-point functions & ghosts** \Rightarrow

$$\square f_i - (-2\pi)f_i = \frac{1}{\sqrt{\det(g - \mathcal{F})}} \partial_\mu (\sqrt{\det(g - \mathcal{F})} G^{\mu\nu} \partial_\nu f_i) - (-2\pi)f_i = 0.$$

= on-shell condition \Rightarrow

$$S = -\frac{1}{2g_o^2} \int d^D x \sqrt{g - \mathcal{F}} \left\{ G^{\mu\nu} \cdot \partial_\mu T \cdot \partial_\nu T - \frac{1}{\alpha'} T \cdot T - \sqrt{\frac{8}{9\alpha'}} T \cdot (T \circ T) \right\} .$$

- **2nd derivative order** and dilaton background: **Kontsevich + gauge term:**

$$f \circ g = f \star g - \frac{1}{24} \Theta^{\mu\rho} \Theta^{\nu\sigma} \partial_\rho \partial_\sigma (\log \sqrt{\det(g + \mathcal{F})} e^{-\phi}) f_\mu g_\nu$$

Green Schwarz String

- Type II **target-superspace** $x^M = (x^m, \theta^\mu, \hat{\theta}^{\hat{\mu}})$ with supersymmetry-transformation

$$\begin{aligned}\delta\theta^\mu &= \varepsilon^\mu, & \delta\hat{\theta}^{\hat{\mu}} &= \hat{\varepsilon}^{\hat{\mu}} \\ \delta x^m &= \varepsilon\gamma^m\theta + \hat{\varepsilon}\gamma^m\hat{\theta}\end{aligned}$$

SUSY-invariant one-forms (supervielbeins) in flat superspace

$$E^A \equiv dx^M E_M^A = \left(\underbrace{dx^a + d\theta\gamma^a\theta + d\hat{\theta}\gamma^a\hat{\theta}}_{\Pi^a}, \quad d\theta^\alpha, \quad d\hat{\theta}^{\hat{\alpha}} \right)$$

- GS-action (in conformal gauge)

$$\begin{aligned}S_{GS} &= \int \frac{1}{2}\Pi_z^a\eta_{ab}\Pi_{\bar{z}}^b + \mathcal{L}_{WZ} \\ \mathcal{L}_{WZ} &= -\frac{1}{2}\Pi_z^a \left(\theta\gamma_a\bar{\partial}\theta - \hat{\theta}\gamma_a\bar{\partial}\hat{\theta} \right) + \frac{1}{2} (\partial\theta\gamma^a\theta) \left(\hat{\theta}\gamma_a\bar{\partial}\hat{\theta} \right) - (z \leftrightarrow \bar{z})\end{aligned}$$

- **Fermionic momenta are constrained** (constraints $d_\alpha = p_\alpha - f_\alpha$)

$$p_{z\alpha} = (\gamma_a\theta)_\alpha \left(\partial x^a - \frac{1}{2}\theta\gamma^a\partial\theta - \frac{1}{2}\hat{\theta}\gamma^a\partial\hat{\theta} \right) = f_\alpha(\theta^\mu, \partial_1 x^m, \partial_1\theta^\mu, p_a)$$

- Constraints are **mixed first (κ -symmetry)/ second class**

$$\{d_{z\alpha}(\sigma), d_{z\beta}(\sigma')\} \propto 2\gamma_{\alpha\beta}^a \Pi_{za} \delta(\sigma - \sigma')$$

Siegel's idea (NPB'93): complete to a (centrally extended) closed algebra

$$\{d_{z\alpha}, \Pi_{za}\} \propto 2\gamma_{a\alpha\beta} \partial\theta^\beta \delta(\sigma - \sigma')$$

$$\{\Pi_{za}, \Pi_{zb}\} \propto \eta_{ab} \delta'(\sigma - \sigma')$$

$$\{d_{z\alpha}, \partial\theta^\beta\} \propto \delta_\alpha^\beta \delta'(\sigma - \sigma')$$

- Same chiral algebra from the following free Lagrangian

$$\begin{aligned} S_{free} &= \int \frac{1}{2} \partial x^m \eta_{mn} \bar{\partial} x^n + \bar{\partial} \theta^\alpha p_{z\alpha} + \partial \hat{\theta}^{\hat{\alpha}} \hat{p}_{\bar{z}\hat{\alpha}} = \\ &= \int \underbrace{\frac{1}{2} \Pi_z^a \eta_{ab} \Pi_{\bar{z}}^b + \mathcal{L}_{WZ}}_{\mathcal{L}_{GS}} + \bar{\partial} \theta^\alpha d_{z\alpha} + \partial \hat{\theta}^{\hat{\alpha}} \hat{d}_{\bar{z}\hat{\alpha}} \end{aligned}$$

Classically coincides with GS for $d_\alpha = \hat{d}_{\hat{\alpha}} = 0$ (still mixed first/second class).

Berkovits Pure Spinor String

- Berkovits (hep-th/0001035): implement $d_\alpha = 0$ in cohomology

$$Q = \oint \lambda^\alpha d_{z\alpha}, \quad \hat{Q} = \oint d\bar{z} \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\bar{z}\hat{\alpha}}$$

d_α not pure first class $\Rightarrow Q^2 = 0$ requires pure spinor constraint $\lambda\gamma^a\lambda = 0$

- Berkovits pure spinor string action in flat background (add only \mathcal{L}_{gh})

$$S_{ps} = \int \underbrace{\frac{1}{2}\Pi_z^a \eta_{ab} \Pi_{\bar{z}}^b}_{\mathcal{L}_{GS}} + \bar{\partial}\theta^\alpha d_{z\alpha} + \partial\hat{\theta}^{\hat{\alpha}} \hat{d}_{\bar{z}\hat{\alpha}} + \mathcal{L}_{gh}$$

$$\mathcal{L}_{WZ} = -\frac{1}{2}\Pi_z^a \left(\theta\gamma_a \bar{\partial}\theta - \hat{\theta}\gamma_a \bar{\partial}\hat{\theta} \right) + \frac{1}{2} (\partial\theta\gamma^a\theta) \left(\hat{\theta}\gamma_a \bar{\partial}\hat{\theta} \right) - (z \leftrightarrow \bar{z})$$

$$\Pi_z^a = \partial x^a + \partial\theta\gamma^a\theta + \partial\hat{\theta}\gamma^a\hat{\theta}$$

$$\mathcal{L}_{gh} = \bar{\partial}\lambda^\alpha \omega_{z\alpha} + \partial\hat{\lambda}^{\hat{\alpha}} \omega_{\hat{\alpha}} + L_{z\bar{z}a}(\lambda\gamma^a\lambda) + \hat{L}_{z\bar{z}a}(\hat{\lambda}\gamma^a\hat{\lambda})$$

- Lagrange multiplier L good enough at classical level; quantization of (λ, ω) is tricky
- PS constraint (first class) generates antighost gauge symmetry $\delta_{(\mu)}\omega_{z\alpha} = \mu_{za}(\gamma^a\lambda)_\alpha$

Type II PS String in General Background

Berkovits&Howe [hep-th/0112160]: deform by vertex; Bedoya&Chandía [hep-th/0609161]: 1-loop;
 Guttenberg'06: type II BRST-transformations

- Curved background: up to field redefinitions **most general renormalizable action** with ghostnumber 0 is (with $G_{MN} = E_M^a e^{2\Phi} \eta_{ab} E_N^b$):

$$\begin{aligned}
 S = & \int \frac{1}{2} \partial x^M (G_{MN}(x) + B_{MN}(x)) \bar{\partial} x^N + \bar{\partial} x^M E_M^\alpha(x) d_\alpha + \partial x^M E_M^{\hat{\alpha}}(x) \hat{d}_{\hat{\alpha}} + \mathcal{T}(x) \\
 & + d_\alpha \mathcal{P}^{\alpha\hat{\beta}}(x) \hat{d}_{\bar{z}\hat{\beta}} + \lambda^\alpha C_\alpha^{\beta\hat{\gamma}}(x) \omega_\beta \hat{d}_{\hat{\gamma}} + \hat{\lambda}^{\hat{\alpha}} \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(x) \hat{\omega}_{\hat{\beta}} d_\gamma + \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}}(x) \omega_\beta \hat{\omega}_{\hat{\beta}} \\
 & + \underbrace{(\bar{\partial} \lambda^\beta + \lambda^\alpha \bar{\partial} x^M \Omega_{M\alpha}^\beta(x))}_{\equiv \nabla_{\bar{z}} \lambda^\beta} \omega_{z\beta} + \underbrace{(\partial \hat{\lambda}^{\hat{\beta}} + \hat{\lambda}^{\hat{\alpha}} \partial x^M \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}(x))}_{\equiv \hat{\nabla}_z \lambda^{\hat{\beta}}} \hat{\omega}_{\bar{z}\hat{\beta}} + L_{z\bar{z}a}(\lambda \gamma^a \lambda) + \\
 & + \hat{L}_{z\bar{z}\hat{a}}(\hat{\lambda} \gamma^{\hat{a}} \hat{\lambda})
 \end{aligned}$$

- A nonconstant **tachyon** background \mathcal{T} will not be BRST-invariant
 (flat: $s\mathcal{T} = \lambda^\alpha \nabla_\alpha \mathcal{T} \stackrel{!}{=} 0 \Rightarrow [\nabla_\alpha, \nabla_\beta] \mathcal{T} = -2\gamma_{\alpha\beta}^a \nabla_a \mathcal{T} \stackrel{!}{=} 0 \Rightarrow \mathcal{T} \stackrel{!}{=} const$)
- The general ansatz for the **BRST-currents** (gh#1, conf weight 1) in the curved background can (reparametrizing d) be brought to

$$\mathbf{j}_z = \lambda^\alpha d_{z\alpha} + \lambda^\alpha W_{\alpha M}(x) \partial_z x^M, \quad \mathbf{j}_{\bar{z}} = 0, \quad (Q = \oint dz \mathbf{j}_z)$$

SUGRA and superspace deformation

- Ooguri, Vafa [hep-th/0303063]: Hybrid formalism, 4d superspace
de Boer, Grassi, Nieuwenhuizen [hep-th/0302078]: 10d, constant RR background
- Berkovits [hep-th/0205154]: (non-abelian) SUSY Born-Infeld from pure spinor (with boundary fermions)
- Most tedious part: solution of constraints, θ -expansion (cf. Dimitrios' talk)

Summary

- Done: Cyclicity through 2nd derivativ order, NC tachyon (BSFT)
- Photon correlation, RR backgrounds ... (in progress)
 - guess correct forms of effective actions (deformation of BI?)
by “non-commutative intuition” ?
gauge and coordinate invariances, Seiberg–Witten maps → photon effective action
deformed superspace (RR backgrounds) and non-abelien Born–Infeld
 - check by computing correlations
 - prove “non-topological formality” and cyclicity ...

Applications

- Kalb-Ramond field B in WZW
- RR-fields (non-anticomm. superspace) AdS/CFT ...
Matseev, Tseytlin '98: $SU(2, 2|4)/SO(4, 1) \times SO(5)$
- Fluxes in internal dimensions: discrete torsion & NC: [Bereinstein, Leigh '01]