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Matter fields in $SO(2,3)_\star$ model of noncommutative gravity

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NC geometry and gravity

Early Universe $\Longrightarrow$ Quantum gravity $\Longrightarrow$ Quantum space-time
Noncommutative (NC) space-time $\Longrightarrow$ Gravity on NC spaces.

General Relativity is based on the diffeomorphism symmetry. This concept (space-time symmetry) is difficult to generalize to NC spaces. Different approaches:

NC spectral geometry [Chamseddine, Connes, Marcolli '07; Chamseddine, Connes, Mukhanov '14].

Emergent gravity [Steinacker '10, '16].

Frame formalism, operator description [Burić, Madore '14; Fritz, Majid '16].

Twist approach [Wess et al. '05, '06; Ohl, Schenckel '09; Castellani, Aschieri '09; Aschieri, Schenkel '14].

NC gravity as a gauge theory of Lorentz/Poincaré group [Chamseddine '01,'04, Cardela, Zanon '03, Aschieri, Castellani '09,'12; Dobrski '16].

Overview

Review of $SO(2,3)_\star$ NC gravity
- General
- Action
- NC Minkowski space-time

Adding matter fields
- Spinors
- $U(1)$ gauge field
- NC Landau problem

Discussion

$SO(2,3)_\star$ NC gravity: General

$SO(2,3)_\star$ NC gravity is based on:

-NC space-time $\rightarrow$ Moyal-Weyl deformation with small, constant NC parameter $\theta^{\alpha\beta} = -\theta^{\beta\alpha}$; $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$.
-gravity $\rightarrow$ $SO(2,3)$ gauge theory with symmetry broken to $SO(1,3)$, [Stelle, West '80; Wilczek '98].
-$\star$-product formalism: Moyal-Weyl $\star$-product.
-Seiberg-Witten (SW) map $\rightarrow$ relates NC fields to the corresponding commutative fields.

Our goals:

-consistently construct NC gravity action, add matter fields, calculate equations of motion and find NC gravity solutions; investigate phenomenological consequences of the constructed model.
-diffeomorphism symmetry broken by fixing constant $\theta^{\alpha\beta}$, give physical meaning to $\theta^{\alpha\beta}$.

$SO(2,3)_\star$ NC gravity: Action

$SO(2,3)_\star$ gauge theory: gauge field ω_μ and field strength tensor $F_{\mu\nu}$ of the $SO(2,3)$ gauge group:

$$\begin{aligned}
\omega_\mu &= \frac{1}{2}\omega_\mu^{AB}M_{AB} = \frac{1}{4}\omega^{ab}\sigma_{ab} + \frac{1}{2}e_\mu^a\gamma_a\,, && (1)\\
F_{\mu\nu} &= \frac{1}{2}F_{\mu\nu}^{AB}M_{AB} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu,\omega_\nu]\\
&= \frac{1}{4}\Big(R_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a)\Big)\sigma_{ab} + \frac{1}{2}F_{\mu\nu}^{a5}\gamma_a\,, && (2)
\end{aligned}$$

with

$$\begin{aligned}
[M_{AB}, M_{CD}] &= i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC})\,, && (3)\\
\eta_{AB} &= (+,-,-,-,+), \quad A,B = 0,\dots,3,5\\
M_{AB} &\to (M_{ab}, M_{a5}) = (\frac{i}{4}[\gamma_a,\gamma_b], \frac{1}{2}\gamma_a),\ a,b = 0,\dots 3\\
R_{\mu\nu}^{ab} &= \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \omega_{\mu c}^a\omega_\nu^{cb} - \omega_{\mu c}^b\omega_\nu^{ca},\\
F_{\mu\nu}^{a5} &= \frac{1}{l}\Big(\nabla_\mu e_\nu{}^a - \nabla_\nu e_\mu{}^a\Big), \quad \nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_{\mu b}^a e_\nu{}^b.
\end{aligned}$$

The decomposition in (1, 2) very much resembles the definitions of curvature and torsion in Einstein-Cartan gravity leading to General Relativity. Indeed, after introducing proper action(s) and the breaking of $SO(2,3)$ symmetry (gauge fixing) down to the $SO(1,3)$ symmetry one obtains this result [Stelle, West '80].

To fix the gauge: the field ϕ transforming in the adjoint representation:

$$\phi = \phi^A \Gamma_A,\ \delta_\epsilon \phi = i[\epsilon, \phi]\,,$$

with $\Gamma^A = (i\gamma_a\gamma_5, \gamma_5)$ and γ_a and γ_5 are the usual Dirac gamma matrices in four dimensions.

Inspired by [Stelle, West '80; Wilczek '98] we define the NC gravity action as

$$S_{NC} = c_1 S_{1NC} + c_2 S_{2NC} + c_3 S_{3NC} \,, \tag{4}$$

with

$$\begin{aligned}
S_{1NC} &= \frac{il}{64\pi G_N} \mathrm{Tr} \int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi} \,, \\
S_{2NC} &= \frac{1}{64\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma} \hat{\phi} \star \hat{F}_{\mu\nu} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} + c.c. \,, \\
S_{3NC} &= -\frac{i}{128\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4x \, \varepsilon^{\mu\nu\rho\sigma} D_\mu \hat{\phi} \star D_\nu \hat{\phi} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi} \,.
\end{aligned}$$

The action is written in the 4-dimensional Minkowski space-time, as an ordianry NC gauge theory. It is invariant under the NC $SO(2,3)_\star$ gauge symmetry and the SW map guarantees that after the expansion it will be invariant under the commutative $SO(2,3)$ gauge symmetry.

Using the SW map solutions for the fields $\hat{F}_{\mu\nu}$ and $\hat{\phi}$ and the Moyal-Weyl $\star$-product, we expand the action (4) in the orders of NC parameter $\theta^{\alpha\beta}$. In the commutative limit $\theta^{\alpha\beta} \to 0$ and after the gauge fixing: $\phi^a = 0, \phi^5 = l$, these actions reduce to

$$
\begin{aligned}
S^{(0)}_{1NC} &\to -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big(\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{\ \ ab}R_{\rho\sigma}^{\ \ cd} + e(R - \frac{6}{l^2})\Big)\,,\\
S^{(0)}_{2NC} &\to -\frac{1}{16\pi G_N}\int \mathrm{d}^4x e\Big(R - \frac{12}{l^2}\Big),\\
S^{(0)}_{3NC} &\to -\frac{1}{16\pi G_N}\int \mathrm{d}^4x e\Big(-\frac{12}{l^2}\Big),
\end{aligned}
$$

with $e_\mu^{\ a} = \frac{1}{l}\omega_\mu^{a5}$, $e = \det e_\mu^a$, $R = R_{\mu\nu}^{\ \ ab}e_a^{\ \mu}e_b^{\ \nu}$. The constants c_1, c_2 and c_3 are arbitrary and can be determined from some consistency conditions.

Comments:

-main adventage of $SO(2,3)$ approach: basic fields are not metric and/or vielbeins but gauge fields of (A)dS group; consequences also in the NC setting.
-after the symmetry breaking: spin connection ω_μ and vielbeins e_μ. They are independent, 1st order formalsim.

-varying (4) with respect to ω_μ and vielbeins e_μ gives equations of motion for these fields. The spin connection in not dynamical (the equation of motion is algebraic, the zero-torsion condition) and can be expressed in terms of vielbeins, 2nd order formalism.

-(4) written in the 2nd order formalism has three terms: Gauss-Bonnet topological term, Einstein-Hilbert term and the cosmological constant term.

-arbitrary constants c_1, c_2 and c_3: EH term requires $c_1 + c_2 = 1$, while the absence of the cosmological constant is provided with $c_1 + 2c_2 + 2c_3 = 0$. Applying both constraints leaves one free parameter (can be used later in the NC generalization).

Calculations show that the first order correction $S_{NC}^{(1)} = 0$. Already known result [Chamseddine '01,'04, Cardela, Zanon '03, Aschieri, Castellani '09].

The first non-vanishing correction is of the second order in the NC parameter; it is long and difficult to calculate. However, the second order corrections can be anayzed sector by sector: high/low energy, high/low/zero cosmological constant, zero/non-zero torsion.

In the low energy sector, i.e., keeping only terms of the zeroth, the first and the second order in the derivatives of vierbeins (linear in $R_{\alpha\beta\gamma\delta}$, quadratic in $T^{a}_{\alpha\beta}$), we calculate NC induced corrections to Minkowski space-time. This calculation cen be generalized to other solutions of vacuum Einstein equations.

$SO(2,3)_\star$ NC gravity: NC Minkowski space-time

A solution of the form:

$$
\begin{aligned}
g_{00} &= 1 - R_{0m0n}x^m x^n \,, \\
g_{0i} &= -\frac{2}{3} R_{0min}x^m x^n \,, \\
g_{ij} &= -\delta_{ij} - \frac{1}{3} R_{imjn}x^m x^n \,,
\end{aligned}
\tag{5}
$$

where $R_{\mu\nu\rho\sigma} \sim \theta^{\alpha\beta}\theta^{\gamma\delta}$: the Reimann tensor for this solution. The coordinates x^μ we started with, are Fermi normal coordinates: inertial coordinates of a local observer moving along a geodesic; can be constructed in a small neighborhood along the geodesic (cylinder), [Manasse, Misner'63; Chicone, Mashoon'06; Klein, Randles '11].

The measurements performed by the local observer moving along the geodesic are described in the Fermi normal coordinates. He/she is the one that measures $\theta^{\alpha\beta}$ to be constant! In any other reference frame, observers will measure $\theta^{\alpha\beta}$ different from constant.
Fixed NC background: gauge fixed diffeomorphism symmetry, prefered reference frame given by Fermi normal coordinates.

Adding matter fields: spinors

Spinors are naturally coupled to gravity in the first order formalism. NC generalization of an action for Dirac spinor coupled to gravity in $SO(2,3)_\star$ model:

$$
\begin{aligned}
\widehat{S}_\psi &= \frac{i}{12}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\, \Big\{ \widehat{\bar\psi} \star (D_\mu\widehat\phi) \star (D_\nu\widehat\phi) \star (D_\rho\widehat\phi) \star (D_\sigma\widehat\psi) \\
&- (D_\sigma\widehat{\bar\psi}) \star (D_\mu\widehat\phi) \star (D_\nu\widehat\phi) \star (D_\rho\widehat\phi) \star \widehat\psi \Big\} \\
&+ \frac{i}{144}\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\widehat{\bar\psi} \star \Big\{ D_\mu\widehat\phi \star D_\nu\widehat\phi \star D_\rho\widehat\phi \star D_\sigma\widehat\phi \star \widehat\phi \\
&- D_\mu\widehat\phi \star D_\nu\widehat\phi \star D_\rho\widehat\phi \star \widehat\phi \star D_\sigma\widehat\phi + D_\mu\widehat\phi \star D_\nu\widehat\phi \star \widehat\phi \star D_\rho\widehat\phi \star D_\sigma\widehat\phi \Big\} \star \widehat\psi + h.c.\,,
\end{aligned}
\tag{6}
$$

where $D_\sigma\widehat\psi = \partial_\sigma\widehat\psi - i\widehat\omega \star \widehat\psi$ is the $SO(2,3)$ covariant derivative in the defining representation. Expanging the action (6) ($\star$-product, SW-map) gives a nontivial first order correction for a Dirac fermion coupled to gravity.

Phenomenological consequences: in the flat space-time limit we find a deformed propagator:

$$iS_F(p) = \frac{i}{\not{p} - m + i\epsilon} + \frac{i}{\not{p} - m + i\epsilon}(i\theta^{\alpha\beta}D_{\alpha\beta})\frac{i}{\not{p} - m + i\epsilon} + \dots \,, \qquad (7)$$

$$D_{\alpha\beta} = \frac{1}{2l}\sigma_\alpha{}^\sigma p_\beta p_\sigma + \frac{7}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5 p_\sigma - (\frac{m}{4l^2} + \frac{1}{6l^3})\sigma_{\alpha\beta} \,. \qquad (8)$$

A spinor moving along the z-axis and with only $\theta^{12} = \theta \neq 0$ has a deformed dispersion relation (analogue to the birefringence effect):

$$\vec{v}_{1,2} = \frac{\partial E}{\partial \vec{p}} = \frac{\vec{p}}{E_{\vec{p}}}\left[1 \pm \left(\frac{m^2}{12l^2} - \frac{m}{3l^3}\right)\frac{\theta}{E^2_{\vec{p}}} + \mathcal{O}(\theta^2)\right] \,, \qquad (9)$$

with $E_{\vec{p}} = \sqrt{m^2 + p_z^2}$. These results are different from the usual NC free fermion action/propagator in flat space-time:

$$\widehat{S}_\psi = \int \mathrm{d}^4x \widehat{\bar{\psi}} \star \left(i\not{\partial}\widehat{\psi} - m\widehat{\psi}\right) = \int \mathrm{d}^4x \widehat{\bar{\psi}}\left(i\not{\partial}\widehat{\psi} - m\widehat{\psi}\right).$$

Adding matter fields: $U(1)$ gauge field

Metric tensor in $SO(2,3)$ gravity is an emergent quantity. Therefeore, it is not possible to define the Hodge dual $*_H$.

Yang-Mills action $S \sim \int F \wedge (*_H F)$ cannot be defined. A method of auxiliary field $\widehat{f}$ [Aschieri, Castellani '12], with $\delta^\star_\varepsilon \widehat{f} = i[\widehat{\Lambda}_\varepsilon \stackrel{\star}{,} \widehat{f}]$. An action for NC $U(1)$ gauge field coupled to gravity in $SO(2,3)_\star$ model:

$$
\begin{aligned}
\widehat{S}_A = &-\frac{1}{16l}\mathrm{Tr}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\Big(\hat{f}\star\hat{\mathbb{F}}_{\mu\nu}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi} \\
&+\frac{i}{3!}\hat{f}\star\hat{f}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi}\Big) + h.c.\,. \qquad (10)
\end{aligned}
$$

After the expansion ($\star$-product, SW-map) and on the equations of motion $f_{a5}=0$, $f_{ab} = -e^\mu_a e^\nu_b \mathcal{F}_{\mu\nu}$ the zeroth order of the action reduces to

$$
S_A = -\frac{1}{4}\int \mathrm{d}^4x\, e\, g^{\mu\rho}g^{\nu\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\,. \qquad (11)
$$

describing $U(1)$ gauge field minimally coupled to gravity.

NC Landau problem

Phenomenological consequences of our model and NC in general: the NC Landau problem: an electron moving in the x-y plane in the constant magnetic field $\vec{B} = B\vec{e}_z$. Our model in the flat space-time limit gives

$$\left(i\partial\!\!\!/ - m + A\!\!\!/ + \theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}\right)\psi = 0\,, \tag{12}$$

where $\theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}$ is given by

$$\begin{aligned}\theta^{\alpha\beta}\mathcal{M}_{\alpha\beta} = \theta^{\alpha\beta}\Big\{ &-\frac{1}{2l}\sigma_\alpha{}^\sigma\mathcal{D}_\beta\mathcal{D}_\sigma + \frac{7i}{24l^2}\epsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\mathcal{D}_\sigma - \left(\frac{m}{4l^2} + \frac{1}{6l^3}\right)\sigma_{\alpha\beta} \\ &+ \frac{3i}{4}\mathcal{F}_{\alpha\beta}\mathcal{D}\!\!\!\!/ - \frac{i}{2}\mathcal{F}_{\alpha\mu}\gamma^\mu\mathcal{D}_\beta - \left(\frac{3m}{4} - \frac{1}{4l}\right)\mathcal{F}_{\alpha\beta}\Big\}.\end{aligned} \tag{13}$$

For simplicity: $\theta^{12} = \theta \neq 0$ and $A_\mu = (0, By, 0, 0)$. Assume

$$\psi = \begin{pmatrix}\varphi(y) \\ \chi(y)\end{pmatrix} e^{-iEt + ip_x x + ip_z z}\,. \tag{14}$$

with φ, χ and E represented as powers series in θ.

Deformed energy levels, i.e., NC Landau levels are given by

$$\begin{aligned} E_{n,s} &= E^{(0)}_{n,s} + E^{(1)}_{n,s}\,, \\ E^{(0)}_{n,s} &= \sqrt{p_z^2 + m^2 + (2n+s+1)B}\,, \\ E^{(1)}_{n,s} &= -\frac{\theta s}{E^{(0)}_{n,s}}\left(\frac{m^2}{12l^2} - \frac{m}{3l^3}\right)\left(1 + \frac{B}{(E^{(0)}_{n,s}+m)}(2n+s+1)\right) + \frac{\theta B^2}{2E^{(0)}_{n,s}}(2n+s+1)\,. \end{aligned} \tag{15}$$

Here $s = \pm 1$ is the projection of electron spin. In the nonrelativistic limit and with $p_z = 0$, (15) reduces to

$$\begin{aligned} E_{n,s} &= m - s\theta\left(\frac{m}{12l^2} - \frac{1}{3l^3}\right) + \frac{2n+s+1}{2m}B_{eff} - \frac{(2n+s+1)^2}{8m^3}B_{eff}^2 + \mathcal{O}(\theta^2)\,, \quad (16) \\ B_{eff} &= (B + \theta B^2)\,. \end{aligned}$$

Consistent with string theory interpretation of noncommutativity as a Neveu-Schwarz B-field.

In addition, the induced magnetic dipole moment of an electon is given by

$$\mu_{n,s} = -\frac{\partial E_{n,s}}{\partial B} = -\mu_B(2n+s+1)(1+\theta B)\,, \tag{17}$$

where $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton.

Some numbers:

-$\theta = \frac{\hbar^2 c^2}{\Lambda_{NC}^2}$ and $\Lambda_{NC} \sim 10\, TeV$,
-accuracy of magnetic moment measurements $\delta\mu_{n,s} \sim 10^{-13}$,
-for observable effects in $\mu_{n,s}$, $B \sim 10^{11}\, T$ needed. This is the magnetic field of some neutron stars (magnetars), in laboratory $B \sim 10^3\, T$.

Discussion

- A consistend model of NC gravity coupled to matter fields.
- Pure gravity:
  -NC as a source of curvature and torsion.
  -The breaking of diffeomorphism invariance is understood as gauge fixing: a prefered refernce system is defined by the Fermi normal coordinates and the NC parameter $\theta^{\mu\nu}$ is constant in that particular reference system.
- Coupling of matter fields: spinors and $U(1)$ gauge field.
  -deformed propagator and dispersion relations for free fermions in the flat space-time limit.
  -nonstandard NC Electrodynamics (QED), new terms compared with the standard QED on the Moyal-Weyl NC space-time: phenomenological consequences, renormalizability.
- Study: corrections to GR solutions (cosmological, Reissner-Nordström black hole,...); fermions in curved space-time (cosmological neutrinos); ...