

BALKAN SCHOOL 2011

PROBLEMS 1

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August 23, 2011

1. **Dirac spinor transformations.** A four-component Dirac spinor transforms under the Lorentz group as

$$\psi \rightarrow S \psi, S = e^{\frac{i}{2} \theta_{\mu\nu} \Sigma^{\mu\nu}}, \Sigma^{\mu\nu} \equiv \frac{1}{4i} [\gamma^\mu, \gamma^\nu], \quad (1)$$

where γ^μ obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (2)$$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (3)$$

- (a) Introduce left and right-handed chiral spinors

$$\psi_{L,R} \equiv \frac{1 \pm \gamma_5}{2} \psi, \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (4)$$

Show that

$$u_{L(R)} \rightarrow e^{-i\sigma/2(\theta \mp i\phi)} u_{L(R)}, \quad (5)$$

with

$$\psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ u_R \end{pmatrix}. \quad (6)$$

What are θ and ϕ ?

- (b) Take a boost in the z direction and find an expression for ϕ .

2. **Parity and charge conjugation.** Define a charge conjugate spinor

$$\psi^c \equiv C \bar{\psi}^T \quad (7)$$

with

$$C^T \gamma^\mu C = -\gamma_\mu^T, C^T = -C, C^\dagger = -C, \quad (8)$$

(we chose explicitly $C = i\gamma_2\gamma_0$).

- (a) Show that

$$\psi^c \rightarrow S \psi^c \quad (9)$$

under the Lorentz group, which shows that ψ^c transforms the same way as ψ , i.e. it is also a proper spinor.

- (b) Take ψ_L and compute its charge conjugate. What is its chirality?

- (c) What happens to u_L and u_R under a parity transformation? Recall that

$$\mathcal{P} : \psi \rightarrow \gamma_0 \psi. \quad (10)$$

3. **Majorana fermions:** Take a left-handed Weyl field

$$\psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix} \quad (11)$$

and construct the following Majorana field

$$\psi_M = \psi_L + C\bar{\psi}_L^T \quad (12)$$

- (a) Show that

$$\psi_L^T C \psi_L \quad (13)$$

is Lorentz invariant.

- (b) See that

$$\bar{\psi}_M \gamma^\mu \partial_\mu \psi_M = 2 \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L, \quad (14)$$

and that the Majorana mass terms can be rewritten as

$$\bar{\psi}_M \psi_M = \psi_L^T C \psi_L + \text{h.c.} \quad (15)$$